4. A General Convergence Theorem.

By Shin-ichi IZUMI.

Mathematical Institute, Tohoku Imperial University, Sendai.

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1. S. Bochner[1] proved the following theorems:

Theorem 1. If $f(\xi)$ is bounded in $(-\infty, +\infty)$ and $K(\xi)$ is absolutely integrable in $(-\infty, +\infty)$, then we have

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f\left(x+\frac{\xi}{n}\right)K(\xi)d\xi=f(x)\int_{-\infty}^{\infty} K(\xi)d\xi. \tag{1}$$

Theorem 2. If (1°) $f(\xi)$ is absolutely integrable in $(-\infty, +\infty)$, (2°) $f(\xi)$ is continuous at $\xi=x$, (3°) $K(\xi)$ is absolutely integrable in $(-\infty, +\infty)$, (4°) $K(\xi)$ is bounded in $(-\infty, +\infty)$ and (5°) $K(\xi)=o(|\xi|^{-1})$ as $|\xi|\to\infty$, then we have (1).

In this paper the following associated theorem is proved:

Theorem 3. If (1°) $\dfrac{f(\xi)}{1+|\xi|}$ and $\dfrac{f^2(\xi)}{1+|\xi|}$ are absolutely integrable in $(-\infty, +\infty)$, (2°) $f(\xi)$ is continuous at $\xi=x$ and (3°) $K(\xi)$ and $\xi K^2(\xi)$ are absolutely integrable in $(-\infty, +\infty)$, then we have (1).

2. We begin with some lemmas.

Lemma 1. If $h(\eta)$ is absolutely integrable in $(-\infty, +\infty)$ and $h(\eta)$ tends continuously to a limit $h(-\infty)$ as $\eta\to-\infty$, then we have

$$\lim_{\nu\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} h(\eta-\nu)\frac{\sin^2\lambda(\xi-\eta)}{\lambda(\xi-\eta)^2}d\eta=h(-\infty), \tag{2}$$

boundedly for any ξ in $(-\infty, +\infty)$, λ being a fixed constant.

Proof. Without loss of generality, we may suppose that $h(-\infty)=0$.

$$\begin{aligned} J&=\int_{-\infty}^{\infty} h(\eta-\nu)\frac{\sin^2\lambda(\xi-\eta)}{\lambda(\xi-\eta)^2}d\eta \\ &=\int_{-\infty}^{\infty} h(\zeta)\frac{\sin^2\lambda(\xi-\zeta-\nu)}{\lambda(\xi-\zeta-\nu)^2}d\zeta \\ &=\int_{-\infty}^{A}+\int_{A}^{\infty} h(\zeta)\frac{\sin^2\lambda(\xi-\zeta-\nu)}{\lambda(\xi-\zeta-\nu)^2}d\zeta \\ &=J_1+J_2, \qquad \text{say.}\end{aligned}$$

1) S. Bochner: Fouriersche Integral, 1933. Cf. T. Takahashi and S. Izumi: Science Reports, Tohoku Univ., 1934.