28. On Hansen's Coefficients in the Expansions for Elliptic Motion.

By Yusuke HAGIHARA.
Astronomical Observatory, Azabu, Tokyo.

(Comm. by K. HIRAYAMA, M.I.A., Mar. 12. 1935.)

Let r be the radius vector, a the semi-major axis, v the true anomaly, ζ the mean anomaly, u the eccentric anomaly, e the eccentricity, and m a positive integer, n an integer, positive or negative. Further put $z=E^{i\zeta}$, where E is the base of Napier's logarithm and $i=\sqrt{-1}$. The coefficients X_j^{nm} in the Laurent expansion of a function:

$$\left(\frac{r}{a}\right)^n E^{imv}=\left(\frac{r}{a}\right)^n(\cos mv+i\sin mv)=\sum_{j=-\infty}^{\infty}X_j^{nm}z^j,$$

are called Hansen's coefficients and were studied by Tisserand[1] with an elementary but complicated analysis. I propose to deduce the same result by a simpler mode of procedure.

The coefficients can be written

$$X_j^{nm}=\frac{1}{2\pi i}\int_C^{(0+)}\left(\frac{r}{a}\right)^n t^m z^{-j-1}dz,$$

where $t=E^{iv}$, by the famous Cauchy's theorem of residues in the theory of analytic functions, the contour of integration being taken so as to make a positive circuit round $z=0$ in the ring-domain excepting $z=0$ and $z=\infty$. Now write $s=E^{iu}$ and

$$\omega=\frac{e}{1+\sqrt{1-e^2}}=\frac{1-\sqrt{1-e^2}}{e}<1,$$

then Kepler's equation can be transformed into

$$z=sE^{-\frac{e}{2}\left(s-\frac{1}{s}\right)}.$$

By the well-known formula for elliptic motion, we have

$$\frac{r}{a}=1-\frac{e}{2}\left(s+\frac{1}{s}\right)=\frac{1}{1+\omega^2}\left(1-\omega s\right)\left(1-\frac{\omega}{s}\right).$$

Hence

$$X_j^{nm}=\frac{1}{2\pi i}\int_{C_s}^{(0+)}\frac{s^m}{(1+\omega^2)^{n+1}}\left(1-\omega s\right)^{n-m+1}\left(1-\frac{\omega}{s}\right)^{n+m+1}$$
$$\times E^{\frac{j\omega}{1+\omega^2}\left(s-\frac{1}{s}\right)}\cdot s^{-j-1}ds,$$

1) F. Tisserand: Traité de Mécanique Céleste. T. 1 (1889), Chap. XV.