52. Displacements in a Manifold of Matrices, II.

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As a continuation of the previous paper[1] we introduce in the present paper some parameter matrices of displacement, which are invariant under the weight variations of a matrix, to which the displacement is to be applied.

1. The weight variation of a matrix $\bar{A}=\rho A$ transforms the parameter matrix $\Gamma(A)$ into

$$\bar{\Gamma}(A)=A\varphi+\Gamma(A), \tag{1}$$

taking $\varphi=d\log\rho$. We consider the new parameter matrix $\wedge(A)$ which has the following form:

$$\wedge(A)=\Gamma(A)+Af(\Gamma, A), \tag{2}$$

where $f(\Gamma, A)$ is a quantity depending on the parameter matrix Γ and A. We assume for the sake of simplicity that $f(\Gamma, A)$, considered as a function of Γ, is regular analytic in the neighbourhood of $\Gamma=0$. As Γ is homogeneous of the first dimension with respect to A, $f(\Gamma(A), A)$ must be homogeneous of zero-th dimension with respect to A. In order that the parameter matrix $\wedge(A)$ may be invariant under transformation (1), it is necessary and sufficient that

$$\bar{\Gamma}(A)+Af(\bar{\Gamma}, A)=\Gamma(A)+Af(\Gamma, A),$$

that is

$$\varphi+f(\Gamma+A\varphi, A)=f(\Gamma, A), \tag{3}$$

for any value of φ. From (3) it follows that $f(\Gamma+A\varphi, A)$ must be a linear function of φ, and that the function $f(\Gamma, A)$ must satisfy the differential equation:

$$\frac{d}{d\varphi}f(\Gamma+A\varphi, A)\equiv\frac{\partial f(\Gamma, A)}{\partial\Gamma}\cdot A=-1. \tag{4}$$

Putting $\Gamma=0$ we moreover get from (3)

$$f(A\varphi, A)=-\varphi+f(0, A),$$

where $f(0, A)$ can be an arbitrary but homogeneous function of zero-th dimension with respect to A and is independent of Γ. For this reason we may now put $f(0, A)=0$ without loss of generality. Then the function $f(\Gamma, A)$ should satisfy the functional equation:

$$\begin{aligned} &f(A\varphi, A)=\varphi f(A, A),\\ &f(\Gamma+A\varphi, A)=f(\Gamma, A)+f(A\varphi, A). \end{aligned} \tag{5}$$

Let $\psi(\Gamma, A)$ be an arbitrary but linear homogeneous function with respect to Γ, and $\psi(A, A)\neq 0$, then the general solution of (5) has the form

$$f(\Gamma, A)=\Phi(\Pi, A)-\frac{\psi(\Gamma, A)}{\psi(A, A)}, \tag{6}$$

where $\Phi(\Pi, A)$ is an arbitrary function but to satisfy the relations

1) A. Kawaguchi: Displacements in a manifold of matrices, I, Proc. **11** (1935), 39–42.