PAPERS COMMUNICATED

9. *Some Theorems on a Cluster-set of an Analytic Function.*

By Kiyoshi NOSHIRO.

Mathematical Institute, Hokkaido Imperial University, Sapporo.

(Comm. by S. KAKEYA, M.I.A., Feb. 12, 1937.)

1. Let $f(z)$ be uniform and meromorphic in a finite connected domain D. We shall first state some notations—$\mathfrak{D}$: the value-set of $f(z)$ in D, F: the boundary set of $\mathfrak{D}$, H: the set of all cluster values[1] at the boundary of D, $\overline{M}$: the closure of M, CM: the complementary set of M. It is evident that $F \subset H \subset \overline{\mathfrak{D}}$ and they are all closed sets. In general the equality $F=H$ does not hold. For example, if we take $w=f(z)=z^2$ and D: $0<\arg z<\dfrac{3\pi}{2}$, $R_1<|z|<R_2$, then $\mathfrak{D}$ is a ring: $R_1^2<|w|<R_2^2$ and H consists of two segments $(-R_2^2, -R_1^2)$, (R_1^2, R_2^2) and two circles $|w|=R_1^2$, $|w|=R_2^2$. Now suppose that $F=H$. Then we see easily that for any value $\alpha \in \mathfrak{D}$, $f(z)$ never takes α at infinite times, for otherwise α would be a cluster value, so that α would belong to $F=H$. This is a contradiction. Next we shall show that $f(z)$ is *exactly p-valent* in D, if a certain value $\alpha \in \mathfrak{D}$ is taken p times. Consider a closed circular domain $\overline{K}$ contained entirely interior to $\mathfrak{D}$. The set of points z, each of which has an image in $\overline{K}$, in general, consists of a finite or an enumerable infinity of connected domains $\overline{\Delta}_i$ in D. However, since $H=F$, each $\overline{\Delta}_i$ must lie completely in the interior of D and so the number of $\overline{\Delta}_i$ is finite. Then $f(z)$ takes in D any value $\alpha \in K$ exactly at the same number of times, say p times, since this holds in each Δ_i by the principle of arguments. Now, let α and β be two finite points in $\mathfrak{D}$. Then we can find a finite sequence of closed circular discs, $\overline{K}_0, \overline{K}_1, \ldots\ldots, \overline{K}_n$ such that each $\overline{K}_i \subset \mathfrak{D}$, $\alpha \in K_0$, $\beta \in K_n$ and $K_i \cdot K_{i+1} \neq 0$ where $i=0, 1, \ldots\ldots, n-1$. Hence $f(z)$ takes α and β at the same number of times, then $f(z)$ is *exactly p-valent* in D, i. e. $f(z)$ *takes in D any value p times.* Conversely, if $f(z)$ is exactly p-valent, then it follows that $H=F$. Let α be an arbitrary finite value in $\mathfrak{D}$ and a_i be an α-point of order p_i. If there are n α-points in total, then clearly $p=\sum_{i=1}^{n} p_i$. Let $\overline{K}_i$ be a small circle: $|z-a_i| \leqq \rho$, lying within D, such that $\overline{K}_j \cdot \overline{K}_{j'}=0$ $(i \neq j')$, and denote by $\mathfrak{D}_i$ the value-set of $f(z)$ in K_i. Then there is a circle C: $|w-\alpha|<\sigma$, contained in $\prod_{i=1}^{n} \mathfrak{D}_i$, any value of which can be taken at least p_i times in each K_i $(i=1, 2, \ldots\ldots, n)$, provided that σ is sufficiently small. Consequently it follows that α cannot be a cluster-value, for otherwise there be a point $z' \in D-\sum_{i=1}^{n} \overline{K}_i$ such

1) We call α a cluster value of $f(z)$ at $z=\zeta$, if there exists a sequence $z_n \to \zeta$, $z_n \neq \zeta$, $z_n \in D$, such that $f(z_n) \to \alpha$.