43. *A Problem Concerning the Second Fundamental Theorem of Lie.*

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(Comm. by T. YOSIE, M.I.A., May 12, 1937.)

§ 1. The problem and the theorem.

Let $\mathfrak{R}$ denote the set of all the matrices of a fixed degree, say n, with complex numbers as coefficients. We introduce a topology in $\mathfrak{R}$ by the *absolute value*

$$|A|=\sqrt{\sum_{i,j=1}^{n}|a_{ij}|^2}, \qquad A=\|a_{ij}\|.$$

If $\mathfrak{G}$, a subset of non-singular matrices $\in \mathfrak{R}$, is a group with respect to the matrix-multiplication, it is a topological group by the distance $|A-B|$.

The topological group $\mathfrak{G}$ is called a *Lie group*, if there exist a finite number, say m, of elements $X_1, X_2, \ldots\ldots, X_m \in \mathfrak{R}$ which satisfy the conditions:

1). $X_1, X_2, \ldots\ldots, X_m$ are linearly independent with real coefficients.

2). $\exp(\sum_{i=1}^{m} t_iX_i) \in \mathfrak{G}$, t real.[1]

3). There exists a positive ε such that any element $A \in \mathfrak{G}$ may be represented uniquely in the form

$$A=\exp(\sum_{i=1}^{m} t_iX_i), \ t \text{ real},$$

if $|A-E| \leqq \varepsilon$ (E the unit-matrix of $\mathfrak{R}$).

By a theorem of J. von Neumann[2] $\mathfrak{G}$ is a Lie group if and only if it is locally compact. Here, for convention, a discrete group is also called a Lie group. If $\mathfrak{G}$ is a Lie group, the set $\mathfrak{J}$ of all the elements $\sum_{i=1}^{m} t_iX_i$, t real, satisfies:

(α). $\mathfrak{J}$ is a real linear space which has a finite base with real coefficients, viz, $X_1, X_2, \ldots\ldots, X_m$.

(β). $[X, Y]=XY-YX \in \mathfrak{J}$ with $X, Y \in \mathfrak{J}$.

$\mathfrak{J}$ is called the *Lie ring* of the Lie group $\mathfrak{G}$, the two ring-operations being the vector-addition and the *commutator-multiplication* $[X, Y]$. It is the set of all the *differential quotients* of $\mathfrak{G}$ at E.[3] The differential quotient of $\mathfrak{G}$ at E is defined by $\lim_{i\to\infty}\big((A_i-E)/\varepsilon_i\big)$, where $A_i(\neq E) \in \mathfrak{G}$ and real ε_i $(\neq 0)$ are such that $\lim_{i\to\infty} A_i=E$, $\lim_{i\to\infty} \varepsilon_i=0$.

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1) $\exp(X)=\sum_{n=0}^{\infty}(X^n/n!)$.

2) See K. Yosida: Jap. J. of Math. **13** (1936), p. 7. Neumann's original statement (M. Z. **30** (1929), p. 3) reads as follows:

$\mathfrak{G}$ is a Lie group if $\mathfrak{G}$ is closed in the group of all the non-singular matrices $\in \mathfrak{R}$.

3) Cf. K. Yosida: loc. cit.