39. *Expansion in Bessel Functions.*

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1. Recently J. Delsarte[1] indicated a formal process of expanding arbitrary function, and proposed the question to discuss its convergence. In this paper we will give some sufficient conditions for the convergence of Bessel expansions under his formulation. The method is that used by us in a memoir on Cauchy's series.[2]

Let $f(x)$ be a function defined over $(0, \omega)$ $(\omega > 1)$ and satisfying the following two conditions that (1°) $x^{2p+1}f(x)$ is Lebesque-integrable in $(0, \omega)$, and that (2°) $\lim_{x\to 0} x\, f(x) = 0$.

Let us put, after Delsarte,[3]

$$j(\lambda x) = \frac{2^p \Gamma(p+1)}{(\lambda x)^p} J_p(\lambda x) \tag{1}$$

and

$$L_\lambda\big(f(x)\big) = \frac{\pi}{2x^p}\int_0^x \xi^{p+1}[Y_p(\lambda x)J_p(\lambda\xi) - J_p(\lambda x)Y_p(\lambda\xi)]f(\xi)d\xi . \tag{2}$$

The section of Delsarte series is, then, given by the contour-integral[4]

$$S_r(x;f) \equiv \frac{1}{2\pi i}\oint_{\mathfrak{C}_r} \frac{-2\lambda j_p(\lambda x)}{\delta[j_p(\lambda x)]}\,\delta\Big[L_\lambda\big(f(x)\big)\Big]d\lambda , \tag{3}$$

where the contour $\mathfrak{C}_r$ is composed of the segment (on the imaginary axis) from $i\rho_r$ to $-i\rho_r$ and the curve $\bar{\mathfrak{C}}_r$ placed in the positive half-plane which starts from $-i\rho_r$ and ends at $i\rho_r$, meeting the real axis at τ_r. Let us designate the parts of $\bar{\mathfrak{C}}_r$ which belong to the first and the fourth quadrants by $\bar{\mathfrak{C}}_r^{(4)}$ and $\bar{\mathfrak{C}}_r^{(1)}$ respectively.

As to the continuity of the linear functionnal δ we assume the following Property (C): if $\{f_n(x)\}$ is an arbitrary sequence of functions everywhere differentiable in the interval $[0, 1]$, and if the convergences of $\{f_n(x)\}$ and $\{f_n'(x)\}$ to their respective limits $f(x)$ and $f'(x)$ happen at each point of the interval $[0, 1]$, then $\delta[f_n(x)]$ tends to $\delta[f(x)]$.

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1) J. Delsarte: (I) Sur un principe générale de developpement des fonctinos d'une variable réelle en série de fonctions entières. C. R. Paris, **200** (1935). (II) Sur l'application d'un principle général de développement des fonctions d'une variable, aux séries de fonctions de Bessel. C. R. Paris, **200** (1935). (III) Sur un procédé de développement des fonctions en séries et sur quelques applications. J. Math. pures appl., IX. s. **15** 97–102 (1936).

2) T. Kitagawa: On the theory of linear translatable functional equation and Cauchy's series. Japanese Journ. Math. **13** (1937) (under press).

3) See Delsarte (III) pp. 97–100.

4) In Certain points this expression is more general than the Delsarte's formulation where $\delta[j_p(\lambda x)]$ has always simple zero-points. *We assume that $\delta[j_p(\lambda x)]$ does not vanish on the whole imaginary axis including the origin.*