80. A Relation between the Length of a Plane Curve and Angles Stretched by it.

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Let us consider a continuous plane curve C, defined by the equation: $x=\varphi(t)$, $y=\psi(t)$ where $\varphi(t)$ and $\psi(t)$ are one-valued and continuous functions of t in $[0,1]=[0\leqq t\leqq 1]$. For brevity, let us call a continuous plane curve without double point a *Jordan curve*. A Jordan curve is therefore a continuous transformation of the segment $[0,1]$ which is biuniform with a possible exception: $\varphi(0)=\varphi(1)$, $\psi(0)=\psi(1)$. We shall denote this transformation by $(x,y)=f(t)$. Let T be a set of numbers contained in $[0,1]$. As t ranges over T, the point (x,y) ranges over a sub-set A of C. A will be called a *generalized arc of* C, and denoted by $f(T)$. The minimum value τ_0 and the maximum value τ_1 of $\overline{T}$[1] are called *extreme values of* A.

Let now C be rectifiable, and let $U=[a,b]$ be an interval contained in $[0,1]$. The arc $f\{U\}$ will be denoted by $\mathfrak{A}(a,b)$. Devide $[a,b]$ into n sub-intervals by the points $t_0=a<t_1<t_2<\cdots\cdots<t_n=b$. To the value $t_m(m=0,1,2,\ldots\ldots,n)$, there corresponds the point $p_m=f(t_m)$ of the curve. Denote by (p_{m-1},p_m) the segment between two points p_{m-1}, p_m, and by $(\overline{p_{m-1},p_m})$ its length. We define as usual the length of the arc $\mathfrak{A}(a,b)$ as the limit of the sum $\sum_{m=1}^{n}(\overline{p_{m-1},p_m})$ (the length of the polygon whose vertices are p_m), when the greatest of the lengths t_m-t_{m-1} tends to 0, and denote it by $l(a,b)$. The complement of $\overline{T}$ (in regard of $[\tau_0,\tau_1]$) is decomposed into at most enumerable infinity of contiguous intervals (a_r,b_r). We call $l(A)=l(\tau_0,\tau_1)-\sum_{r=1}^{\infty}l(a_r,b_r)$ *the length of the generalized arc* A.[2]

If $A_1,A_2,\ldots\ldots,A_k$ are a finite number of generalized arcs of C such as $A=\sum_{\nu=1}^{k}A_\nu$, we have, as usual

$$l(A)\leqq l(A_1)+l(A_2)+\cdots\cdots+l(A_k).$$

We can divide the interval $[\tau_0,\tau_1]$ into n sub-intervals by the points of T: $t_0<t_1<t_2<\cdots\cdots<t_n$, so that the sum $\sum_{m=1}^{n}(\overline{p_{m-1},p_m})$ converges to $l(\tau_0,\tau_1)-\sum_{r=1}^{\infty}l(a_r,b_r)+\sum_{r=1}^{\infty}\left(\overline{f(a_r),f(b_r)}\right)=l(A)+\sum_{r=1}^{\infty}\left(\overline{f(a_r),f(b_r)}\right)$ as n tends to infinity.

Theorem 1. Let A be a generalized arc of a rectifiable Jordan curve C, whose extreme values are τ_0, τ_1. Suppose that A is contained in a circle of radius $S(S>0)$, and that to each point p of A, we can associate two half-lines pq, pq' with the properties:

1) $\overline{T}$ is the set of all points of T together with its limiting points.

2) $l(A)$ is the linear measure of $\overline{A}$ on C.