102. *Notes on Fourier Series (I): Riemann Sum.*

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1. Let $f(x)$ be a periodic function with period 1 and let us write

(1) $$f_k(x)=\frac{1}{k}\sum_{\nu=0}^{k-1} f\left(x+\frac{\nu}{k}\right).$$

If $f(x)$ is integrable in the Riemann sense, then

(2) $$\lim_{k\to\infty} f_k(x)=\int_0^1 f(t)dt.$$

Jessen[1] has shown that if $f(x)$ is integrable (in the Lebesgue sense), then

$$\lim_{n\to\infty} f_{2^n}(x)=\int_0^1 f(t)dt$$

for almost all x. Ursell[2] has shown that (2) is not necessarily true for integrable function $f(x)$ for almost all x, and (2) holds almost everywhere when $f(x)$ is positive decreasing and of squarely integrable in (0,1).

The object of the prpesent paper is to prove the following theorem.

Theorem. Let $f(x)$ be integrable and

(3) $$f(x)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos 2\pi nx+b_n\sin 2\pi nx).$$

If $a_n\sqrt{\log n}$ and $b_n\sqrt{\log n}$ are Fourier coefficients of an integrable function, then (2) holds almost everywhere.

For the validity of (2) almost everywhere $f(x)$ can be discontinuous in a null set, for the condition of the theorem depends on the Fourier coefficients of $f(x)$ only. The condition of the theorem is satisfied when

$$\sum_{n=2}^{\infty}(a_n^2+b_n^2)\log n<\infty.$$

In this case, by the Riesz-Fischer theorem $a_n\sqrt{\log n}$ and $b_n\sqrt{\log n}$ are Fourier coefficients of squarely integrable function and then of integrable function.

2. Let us write

$$c_0=\frac{1}{2}a_0;\qquad c_n=\frac{1}{2}(a_n-ib_n),\qquad c_{-n}=\bar{c}_n\qquad (n>1),$$

1) Jessen, Annals of Math., **34** (1934).
2) Ursell, Journ. of the London Math. Soc., **12** (1937).