12. On a Theorem of K. Löwner on Univalent Functions.

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(Comm. by S. KAKEYA, M.I.A., Feb. 12, 1938.)

1. We denote by (S) the family of normalized univalent functions

$$f(z)=z+a_2z^2+a_3z^3+\cdots\cdots, \tag{1}$$

regular for $|z|<1$, and by (S_k), $k=2, 3, \ldots\ldots$, the family of normalized univalent functions given by

$$f_k(z)=f^{\frac{1}{k}}(z^k)=z+a_{k+1}^{(k)}z^{k+1}+a_{2k+1}^{(k)}z^{2k+1}+\cdots\cdots. \tag{2}$$

Löwner[1] succeeded to prove that, for any function of (S),

$$|a_3|\leqq 3, \tag{3}$$

which relates to the Bieberbach's conjecture, using his own essential theorem of expressing all coefficients of any function $f(z)$ of (S) by parametric function corresponding to $f(z)$ itself.

On the other hand, Fekete and Szegö[2] have recently proved that, for $k=2, 3, \ldots\ldots$,

$$|a_{2k+1}^{(k)}|\leqq\frac{2}{k}e^{-2[(k-1)/(k+1)]}+\frac{1}{k}. \tag{4}$$

It is remarkable for us that the inequality (4) becomes for $k=2$,

$$|a_5^{(2)}|\leqq e^{-\frac{2}{3}}+\frac{1}{2}>1, \tag{5}$$

and the extremal case of (4) and (5) are attained by a function which is different from the well known extremal function

$$\frac{z}{(1-z^k)^{2/k}}, \qquad (k=2, 3, \ldots\ldots),$$

by which the Bieberbach's conjecture for the class of functions (S) is realized for $k=1$.

In the present note, a coefficient problem closely related to the above mentioned theorems of Löwner, Fekete and Szegö is investigated.

2. Instead of the class (S), Löwner considered the class (S') of univalent functions of the form

$$s(z, t)=e^{-t}\left(z+b_1(t)z^3+b_2(t)z^3+\cdots\cdots\right), \qquad t\geqq 0, \tag{6}$$

1) Math. Annalen, **89** (1923), 103-121. .

2) Jour. London Math. Soc., **8** (1933), 85-89.