PAPERS COMMUNICATED

9. *On the Uniqueness of Haar's Measure.*

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1. For a topological group G, which is locally compact and separable, the uuiqueness[1] of Haar's left-invariant measure is proved by J. v. Neumann.[2] Although the method used by him is very interesting and powerful, his proof is somewhat long. The notion of right-zero-invariance is not necessary for the proof. In this paper we shall give a simplified proof. The essential improvement consists in the adoption of the *right*-invariant measure in the second group, in constructing the measure of the topological product $G \times G$. Since the separability plays no essential rôle in our proof, it can also be, by slight modifications, applied to the case of a non-separable group (the case of a locally bicompact topological group, which is treated by A. Weil[3]), and moreover we can prove, in the same manner, the theorem of the uniqueness of Haar's measure even for the case, when the field G is no longer a topological group, that is, G is simply a topological space S, and when the transitive group G of homeomorphic transformations of S on itself is given.[4]

2. We begin with some definitions:

Given a topological space S and a group G of homeomorphic transformation σ of S on itself, a set function[5] μ of S (not necessarily non-negative) is called *G-invariant* if for any Borel set E of S and for any $\sigma \in G$ we have $\mu(\sigma E) = \mu(E)$. A Borel set E of S is called *G-μ-invariant* if $\mu(E \Delta \sigma E) = 0$[6] for any $\sigma \in G$, and G is called *ergodic on S*, if for any G-invariant totally additive non-negative set function[7] μ and for any G-μ-invariant Borel set E of S we have either $\mu(E) = 0$ or $\mu(G - E) = 0$.[8] In the special case, when S and G coincide, that is, when S is a topo-

1) We shall understand in the following by "uniqueness" always "uniqueness up to a constant factor."

2) J. v. Neumann: The uniqueness of Haar's measure, Recueil Math., **1 (43)** (1936).

3) A. Weil: Sur les groupes topologiques et les groupes mesurés, C. R. **202** (1936).

4) Cf. J. v. Neumann: On the uniqueness of invariant Lebesgue measure, Bull. Amer. Math. Soc., **42** (1936) (Abstract).

5) In this paper we consider only those set functions, which are defined and are finite for all Borel sets whose closure is bicompact. (If the space is separable, the notion of compactness and that of bicompactness coincide.) If μ is non-negative, $\mu(E)$ is defined for any non-bicompact Borel set as the least upper bound of all $\mu(F)$, where F is a bicompact Borel set $\subset E$. $\mu(E)$ might be infinite in this case.

6) We denote by $A \Delta B$ the symmetric difference $A + B - A \cdot B$ of two sets A and B. This is the sum of two sets in Boolean sense.

7) We do not assume that $\mu(U) > 0$ for open set U.

8) Cf. J. v. Neumann and F. J. Murray: On rings of operators, Annals of Math., **37** (1936), p. 195.