61. *Two Fixed-point Theorems Concerning Bicompact Convex Sets.*

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1. In this paper we are concerned with two kinds of fixed-point theorems, which are not essentially new. Theorem 1 is already given by A. Markov[1] and Theorem 2 is a generalisation of a result of J. v. Neumann[2] and W. Maak.[3] But the simple way of proving Theorem 1 and the general formulation of Theorem 2 may be regarded with some interest. It is also to be noticed, that, in spite of their similar appearance, these theorems are treated in entirely different ways. Moreover, we shall give some applications of these theorems, which illustrate the importance of these fixed-point theorems. Only the results and the brief summary of their proofs are given, the details being left to a subsequent paper, which will be published elsewhere.

2. Let B be a convex set in some linear space E, and Γ a family of transformations $\varphi(x)$ of B into itself. $\varphi(x)$ is called to be *affine* if for any $x, y \in B$ and $\lambda, \mu \geqq 0$, $\lambda+\mu=1$, we have $\varphi(\lambda x+\mu y)=\lambda\varphi(x)+\mu\varphi(y)$; and Γ is called to be *abelian* if for any $\varphi, \psi \in \Gamma$ and $x \in B$ we have $\varphi\big(\psi(x)\big)=\psi\big(\varphi(x)\big)$.

Theorem 1. Let B be a non-vacuous, convex, bicompact subset of a locally convex linear topological space E, and let Γ be an abelian family of continuous affine transformations $\varphi(x)$ of B into itself; then there is a point $x \in B$ such that we have $\varphi(x)=x$ for any $\varphi \in \Gamma$.

A. Markov's original proof uses the fixed-point theorem of A. Tychonoff.[4] Since this theorem is a generalisation of Brouwer's fixed-point theorem which is valid for general (not necessarily affine!) continuous transformations, a direct proof will be desirable.

To prove Theorem 1 we proceed as follows: Consider the totality Γ^* of all the transformations $\varphi^*(x)$ of the form:

$$\varphi^*(x)=\frac{1}{n}\left(x+\varphi(x)+\cdots+\varphi^{n-1}(x)\right),$$

$n=1, 2, \dots$; $\varphi \in \Gamma$, where $\varphi^k(x)$ denotes the k-th iterate of $\varphi(x)$. $\varphi^*(x)$ is also a continuous affine transformation of B into itself and Γ^* is abelian. It will then be easy to see that $B_1 \equiv \prod_{\varphi^* \in \Gamma^*} \varphi^*(B)$ is not empty and that any point $x \in B_1$ is a desired fixed-point.

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1) A. Markov: Quelques théorèmes sur les ensembles abéliens, C. R. URSS, **2** (1936), p. 311.

2) J. v. Neumann: Almost periodic function in groups I, Trans. Amer. Math. Soc., **36** (1934), p. 445.

3) W. Maak: Eine neue Definition der fastperiodischen Funktionen, Abh. math. Semin. Hansisch. Univ., **11** (1936), p. 240.

4) A. Tychonoff: Ein Fixpunktsatz, Math. Ann., **111** (1935), p. 767.