# 74. Mean Ergodic Theorem in Banach Spaces.

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## § 1. *Introduction and the theorem.*

The mean ergodic theorem of J. von Neumann reads as follows:

*Let $T$ be a unitary operator in the Hilbert space $\mathfrak{H}$. Then, for any $x \in \mathfrak{H}$, the sequence*

$$(1) \qquad x_n = \frac{T\cdot x + T^2\cdot x + \cdots + T^n\cdot x}{n} \qquad (n=1, 2, \ldots)$$

*converges strongly to a point* $\in \mathfrak{H}$.

Neumann's proof is based upon Stone's theorem concerning the one-parameter group of unitary operators in $\mathfrak{H}$. We find F. Riesz's elementary proof in E. Hopf's book.[1]

Recently C. Visser[2] gave the following theorem:

*Let a linear operator $T$ in $\mathfrak{H}$ satisfy the condition; $\|T^n\| \leqq a$ constant for $n=1, 2, \ldots$. Then, for any $x \in \mathfrak{H}$, the sequence* (1) *converges weakly to a point* $\in \mathfrak{H}$.

He also showed that the mean ergodic theorem is easily obtained from this theorem. Thus we have another elementary proof of the mean ergodic theorem.

In the present note I intend to give a more general

*Theorem. Let a linear operator $T$ in the (real or complex) Banach space $\mathfrak{B}$ satisfy the two conditions:*

(2) $\|T^n\| \leqq$ *a constant $C$ for $n=1, 2, \ldots$,*

(3) *$T$ is weakly completely continuous, viz. $T$ maps the unit sphere $\|x\| \leqq 1$ of $\mathfrak{B}$ on the point set which is weakly compact in $\mathfrak{B}$.*[3]

*Then, for any $x \in \mathfrak{B}$, the sequence* (1) *converges strongly to a point $\bar{x} \in \mathfrak{B}$. We have $T\cdot\bar{x} = \bar{x}$.*

As the existence of the inverse $T^{-1}$ of $T$ is not assumed, this theorem may be applied in the problem of the temporally homogeneous stochastic process.[4] The applications will be published elsewhere.

I here express my hearty thanks to S. Kakutani who kindly communicated me that Visser's weak convergence theorem can be extended to $\mathfrak{B}$.[5]

1) Ergodentheorie, Berlin (1937), 23.

2) Proc. Amsterdam Acad. 16, 5 (1938), 487–495.

3) It is sufficient to assume that, for any $x \in \mathfrak{B}$, the sequence (1) is weakly compact in $\mathfrak{B}$. See the proof below. As the Hilbert space is weakly compact locally such conditions are not needed in Visser's theorem.

4) Cf. my preceding paper.

5) See the following paper of Kakutani, where we find his ingeneous arguments.