84. *Application of Mean Ergodic Theorem to the Problems of Markoff's Process.*

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§ 1. *Two supplements to the Mean Ergodic Theorem.*

Mean Ergodic Theorem. Let $\mathfrak{B}$ *be a (real or complex) Banach space, and denote by* T *a linear operator which maps* $\mathfrak{B}$ *in itself. If*

(1) *there exists a constant* C *such that* $\|T^n\| \leqq C$ *for* $n=1, 2, \dots,$
and

(2) $\begin{cases} \text{for any } x \in \mathfrak{B} \text{ the sequence } x_n = \frac{1}{n}(T+T^2+\cdots+T^n)x \ (n=1, \\ 2, \dots) \text{ is weakly compact in } \mathfrak{B}, \end{cases}$

then

(3) $\begin{cases} \text{there exists a linear operator } T_1, \text{ which maps } \mathfrak{B} \text{ in itself, such} \\ \text{that } \lim_{n\to\infty} \frac{1}{n}(T+T^2+\cdots+T^n)x = T_1x \text{ strongly for any } x \in \mathfrak{B}, \text{ and} \\ TT_1 = T_1T = T_1^2 = T_1. \end{cases}$

T_1 is a projection operator which maps $\mathfrak{B}$ on the proper space $\mathfrak{B}_1$ of T belonging to the proper value 1. Because of (1), (2) is surely satisfied if T is *weakly completely continuous*, viz., if T maps the unit sphere $\|x\| \leqq 1$ of $\mathfrak{B}$ on a point set weakly compact in $\mathfrak{B}$. These results were obtained in our previous notes.[1] We now prove the

Theorem 1. (2) *and hence* (3) *hold good if* T *satisfies* (1) *and if*

(2′) $\begin{cases} \text{there exist an integer } k \text{ and a weakly completely continuous} \\ \text{linear operator } V, \text{ which maps } \mathfrak{B} \text{ in itself, such that} \\ \|T^k - V\| < 1. \end{cases}$

Proof:[2] It is sufficient to prove the case $k=1$. Put $\|T-V\| = a < 1$ and $x_{n,p} = \frac{1}{n}(T+T^2+\cdots+T^p)x$ $(n, p=1, 2, \dots)$. We have $T^p = V_p + D_p$, where $V_p = T^p - (T-V)^p$ is weakly completely continuous with $V_1 \equiv V$ and $\|D_p\| \leqq a^p$. Hence $x_n = x_{n,p} + T^p x_{n,n-p} = x_{n,p} + V_p x_{n,n-p} + D_p x_{n,n-p}$. Since $\|x_{n,n-p}\| \leqq C \cdot \|x\|$ for $n=1, 2, \dots$, there exists (for each p) a subsequence $\{n'\}$ of $\{n\}$ such that $\{V_p x_{n',n'-p}\}$ converges weakly to a point $y_p \in \mathfrak{B}$. Consequently we have (since $\lim_{n'\to\infty} |f(x_{n',p})| = 0$)

$$(4) \qquad \overline{\lim_{n'\to\infty}} |f(x_{n'}) - f(y_p)| \leqq \lim_{n'\to\infty} |f(x_{n',p})| + \overline{\lim_{n'\to\infty}} |f(D_p x_{n',n'-p})| \leqq a^p \cdot \|f\| \cdot C \cdot \|x\|$$

for any linear functional f on $\mathfrak{B}$.

1) K. Yosida: Mean Ergodic Theorem in Banach spaces, Proc. **14** (1938), 292.
S. Kakutani: Iteration of linear operations in complex Banach spaces, ibd., 295.

2) Cf. the arguments given by one of us. See the paper of S. Kakutani cited in (1).