91. *Operator-theoretical Treatment of the Markoff's Process.*

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(Comm. by T. TAKAGI, M.I.A., Dec. 12, 1938.)

§ 1. *Introduction.* Let $P(x, E)$ denote the transition probability that the point x of the interval $(0, 1)$ is transferred, by a simple Markoff's process, into the Borel set E on the interval $(0, 1)$ after the elapse of a unit time. We assume that $P(x, E)$ is completely additive for Borel sets E if x is fixed and that $P(x, E)$ is Borel measurable in x if E is fixed. Then the transition probability after the elapse of n units of time is given by $P^{(n)}(x, E) = \int_0^1 P^{(n-1)}(x, dy)P(y, E)$, $\left(P^{(1)}(x, E) = P(x, E)\right)$.

Under certain general condition given below, W. Doeblin[1] investigated the asymptotic behaviour of $P^{(n)}(x, E)$ for large n. His method of proof is based upon set-theoretical considerations. It may be termed as a direct method. In the present note I intend to give an operator-theoretical treatment of the problem, by virtue of the results of the preceding notes.[2] Our method of proof will make clear the spectral properties of the Markoff's process in question, and the results obtained is somewhat more precise than that of Doeblin. The author is indebted to S. Kakutani in the proof of the lemma 1 and 5 below. I want to express my hearty thanks to him.

§ 2. *Preliminary lemmas.* By definition we have

(1) $P^{(n)}(x, E) \geqq 0$ and $P^{(n)}(x, \Omega) \equiv 1$, where $\Omega =$ the interval $(0, 1)$.

We make on $P(x, E)$ the following assumptions due to Doeblin:

$$(2) \quad \begin{cases} \text{there exist an integer } s \text{ and positive } b, \eta \ (<1) \text{ such that,} \\ \text{if } \operatorname{mes}(E) < \eta, \quad P^{(s)}(x, E) < 1 - b \quad \text{uniformly in } x. \end{cases}$$

Then it is easy to see that

$$(2)' \quad \text{if } \operatorname{mes}(E) < \eta, \quad P^{(t)}(x, E) < 1 - b \quad \text{uniformly in } x \text{ and } t \geqq s.$$

We may decompose $P^{(s)}(x, E)$ as follows[3]:

$$(3) \quad \begin{cases} P^{(s)}(x, E) = \int_E q(x, y)dy + R(x, E), \\ 0 \leqq q(x, y) \leqq \dfrac{1}{\eta}, \quad 0 \leqq R(x, E) < 1 - b. \end{cases}$$

1) W. Doeblin: Sur les propriétés asymptotiques de mouvements régis par certains types de chaines simples, Bull. math. de la Soc. Roumaine des Sciences, **39** (1937), (2), 3–61.

2) K. Yosida: Abstract integral equations and the homogeneous stochastic process, Proc. **14** (1938), 286. K. Yosida and S. Kakutani: Application of mean ergodic theorem to the problem of Markoff's process, ibid. 333. K. Yosida, Y. Mimura and S. Kakutani: Integral operator with bounded kernels, ibid. 359. These notes will respectively be referred to as [I], [II] and [III].

3) [II], the proof of Theorem 7′.