PAPERS COMMUNICATED

90. *Integral Operator with Bounded Kernel.*

By Kôsaku YOSIDA, Yukio MIMURA and Shizuo KAKUTANI.
Mathematical Institute, Osaka Imperial University.
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§ **1.** Let $K(x, y)$ be bounded and measurable in the square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$. Consider the integral operator K which transforms the Banach space (L)[1] in (L).

$$(1) \qquad f \to Kf = g: \quad g(y) = \int_0^1 f(x)\, K(x, y)\, dx\,.$$

It is to be noted that such an operator is not always completely continuous[2] in (L). This may be shown by an example (§ 3). We can, however, prove the following

Theorem 1. Let $N(x, y)$ and $K(x, y)$ be bounded and measurable in $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$. Then the integral operator P defined by the bounded Kernel $P(x, y) = \int_0^1 N(x, z)\, K(z, y)\, dz$ is completely continuous as an operator which maps (L) in (L).

Remark. The integral operator (1) may also be considered as a linear operator which maps (L) in (M),[3] (M) in (M) or (M) in (L).

Proof of Theorem 1: Denote by N and K the integral operators which correspond to the kernels $N(x, y)$ and $K(x, y)$ respectively. P may be considered as a combination of two operators N and K performed successively in this order, where N is an operator which maps (L) in (M) and K is the one which maps (M) in (L): $f \in (L) \to Nf = g \in (M) \to Kg (= Pf) = h \in (L)$.

The unit sphere $\|f\|_L \leqq 1$ of (L) is mapped by N on a set contained in the sphere $\|g\|_M \leqq n$ of (M), where $n = \underset{0 \leq x, y \leq 1}{\text{l. u. b.}} |N(x, y)|$. Hence it is sufficient to prove the

Theorem 2. The integral operator K with bounded kernel $K(x, y)$ is completely continuous as an operator which maps (M) in (L).

Proof: We extend the definition domain of $K(x, y)$ to the infinite square $-\infty < x < +\infty$, $-\infty < y < +\infty$, by putting $K(x, y) = 0$ if the point (x, y) is outside the square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$. Let $Kg = h$, where $g \in (M)$, $\|g\|_M \leqq 1$. By Fubuni-Tonelli's theorem, we have

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1) (L) is the space of all the measurable functions $f(x)$ which are absolutely integrable in $0 \leqq x \leqq 1$. For any $f \in (L)$, we define its norm by $\|f\|_L = \int_0^1 |f(x)|\, dx$.

2) A linear operator which maps the Banach space E_1 in another Banach space E_2 is called to be completely continuous if it maps the unit sphere $\|x\| \leqq 1$ of E_1 on a compact (in E_2) set of E_2.

3) (M) is the space of all the bounded measurable functions defined in $0 \leqq x \leqq 1$. For any $f \in (M)$ we define its norm by $\|f\|_M = \underset{0 \leq x \leq 1}{\text{ess. max.}} |f(x)|$.