33. *Mean Ergodic Theorem in Abstract (L)-Spaces.*

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Let us consider an abstract (L)-space (notation: (AL)). Under abstract (L)-space (AL) we mean a semi-ordered Banach space, whose norm is additive on positive elements. That is, (AL) is a Banach space, which satisfies the following axioms[1] (we denote by $x, y, \dots$ the elements of (AL), by $\|x\|, \|y\|, \dots$ their norm and by λ a real number):

(1) a semi-order relation $x > y$ is defined on (AL), and (AL) is a lattice with respect to this semi-ordering. That is:

(1–1) $x > y$ and $y > z$ imply $x > z$,

(1–2) $x > y$ and $\lambda > 0$ imply $\lambda x > \lambda y$,

(1–3) $x > y$ implies $x+z > y+z$ for any z,

(1–4) $x_n > y_n$, $x_n \to x$ (strongly) and $y_n \to y$ (strongly) imply $x \geqq y$ ($x \geqq y$ means $x > y$ or $x = y$),

(1–5) to any pair of elements x and y, there exists a maximum $z = x \vee y$ such that $z \geqq x$, $z \geqq y$ and $z \leqq z'$ for any z' with $z' \geqq x$, $z' \geqq y$,

(1–6) to any pair of elements x and y, there exists a minimum $w = x \wedge y$ such that $w \leqq x$, $w \leqq y$ and $w \geqq w'$ for any w' with $w' \leqq x$, $w' \leqq y$.

(2) norm is additive on positive elements: $x > 0$ and $y > 0$ imply $\|x+y\| = \|x\| + \|y\|$.

Such a space was discussed by Garrett Birkhoff.[2] He has introduced an abstract (L)-space as a generalization of a concrete (L)-space[3] (notation: (L)), and has discussed the iteration of positive[4] bounded linear operations in such an abstract (L)-space.[5]

The main result of G. Birkhoff may be stated as follows:

Theorem. Let T be a positive bounded linear operation which maps an abstract (L)-space (AL) into itself. If T is of norm 1, and if for any $x \in (AL)$ the sequence $\{T^n(x)\}$ $(n = 1, 2, \dots)$ is bounded from

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1) Separability is not assumed.

2) G. Birkhoff: Dependent probabilities and the space (L), Proc. Nat. Acad. U. S. A., **24** (1938), 154–159.

3) Under concrete (L)-space, we mean the ordinary Banach space (L), composed of all the measurable functions which are absolutely integrable in $0 \leqq x \leqq 1$. For any $f(x) \in (L)$, we define its norm by $\|f\| = \int_0^1 |f(x)|\, dx$. (L) is separable in this norm.

4) A linear transformation, which maps a semi-ordered linear space into itself, is called to be positive if $x \geqq 0$ implies $Tx \geqq 0$.

5) This may be considered as the most general formulation of Markoff's process, since, as is well known, the problem of Markoff's process is nothing but the investigation of the behaviour of the iteration of positive bounded linear operations of norm 1 in the concrete (L)-space (L) (or the space (V). See the footnote (3) on page 122).