44. *Weak Topology and Regularity of Banach Spaces.*

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(Comm. by T. TAKAGI, M.I.A., June 12, 1939.)

1. *Introduction.* Let E be a Banach space (element x with norm $\|x\|$), and let $\bar{E}$ be its conjugate space (i. e. the space of all bounded linear functionals $f(x)$ defined on E, with norm $\|f\|=\underset{|x|\leq 1}{\text{l. u. b.}}\,|f(x)|$). $\bar{E}$ is also a Banach space and its conjugate space $\bar{\bar{E}}$ may be considered. E is called to be *regular* if we have $E=\bar{\bar{E}}$, or equivalently, if every bounded linear functional $X_0(f)$ defined on $\bar{E}$ may be represented in the form: $X_0(f)=f(x_0)$ for any $f\in\bar{E}$, where x_0 is a point from E.

We shall give, in the first part of this paper (§§ 2 and 3), some conditions for the regularity of E. This problem was investigated by several authors and many interesting results were obtained. Our principal idea is to use the *weak topologies* in E and in $\bar{E}$. It seems to me that too little attention has been paid to the weak topologies in Banach spaces, while on the contrary in the theory of Hilbert space weak topology plays an essential rôle. It will be shown in this paper how weak topologies are successfully introduced into such problems. We shall state only the definition of weak topologies and a few fundamental theorems, the rest being left to another occasion.

In the second part of this paper (§ 5), we shall prove that every uniformly convex Banach space is regular (Theorem 3).[1] From this follows easily that Mean Ergodic Theorem holds true in uniformly convex Banach spaces. A direct proof of this fact was obtained recently by Garrett Birkhoff.[2]

The proof of Theorem 3 relies on Helly's theorem.[3] Mr. Yukio Mimura has kindly informed me of a simple and elegant proof of this theorem. This proof is given in § 4. I express my hearty thanks to Mr. Yukio Mimura.

2. *Weak topologies.* (1) *Weak topology in E.* For any $x_0\in E$, its weak neighbourhood $U_1(x_0, f_1, f_2, \dots, f_n, \varepsilon)$ is defined as the totality of all the points $x\in E$ such that $|f_i(x)-f_i(x_0)|<\varepsilon$ for $i=1, 2, \dots, n$, where $\{f_i(x)\}$ $(i=1, 2, \dots, n)$ is an arbitrary system of bounded linear functionals defined on E and $\varepsilon>0$ is an arbitrary positive number.

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1) This theorem was recently proved by D. Milman in a different way. D. Milman: On some criteria for the regularity of spaces of type (B), C. R. URSS, **20** (1938), 243–246. D. Milman's proof uses the notion of transfinite closedness. It is our purpose to avoid, as far as possible, the use of transfinite method in the theory of Banach spaces.

2) G. Birkhoff: The mean ergodic theorem, Duke Math. Journ., **5** (1939), 19–20.

3) E. Helly: Über Systeme linearer Gleichungen mit unendlich vielen Unbekannten, Monatsh. für Math. und Phys., **31** (1921), 60–91.