PAPERS COMMUNICATED

43. Birkhoff's Ergodic Theorem and the Maximal Ergodic Theorem.

By Kôsaku YOSIDA and Shizuo KAKUTANI.
Mathematical Institute, Osaka Imperial University.
(Comm. by T. TAKAGI, M.I.A., June 12, 1939.)

1. *Statement of the theorem.* Let S be a space in which a measure of Lebesgue type is defined, and let T be a one-to-one measure-preserving transformation of S into itself. We do not assume that the total measure mes (S) is finite. For any real valued function $f(x)$ defined on S, we define the functions $\bar{f}(x), \underline{f}(x), f^*(x)$ and $f_*(x)$ as follows:

$$\begin{cases} \bar{f}(x)=\overline{\lim_{n\to\infty}}\dfrac{1}{n}\sum_{i=0}^{n-1}f(T^ix)\,, & \underline{f}(x)=\underline{\lim_{n\to\infty}}\dfrac{1}{n}\sum_{i=0}^{n-1}f(T^ix)\,, \\ f^*(x)=\underset{0\leq n<\infty}{\text{l. u. b.}}\dfrac{1}{n}\sum_{i=0}^{n-1}f(T^ix)\,, & f_*(x)=\underset{0\leq n<\infty}{\text{g. l. b.}}\dfrac{1}{n}\sum_{i=0}^{n-1}f(T^ix)\,. \end{cases}$$

If $f(x)$ is measurable and absolutely integrable on S, then we can prove the following two theorems:

Theorem 1. For any pair of real numbers α and β, we have

$$(1)\quad \begin{cases} \alpha \text{ mes}\big(E(\alpha,\beta)\big) \leqq \displaystyle\int_{E(\alpha,\beta)} f(x)\,dx \leqq \beta \text{ mes}\big(E(\alpha,\beta)\big)\,, \\ \text{where}\quad E(\alpha,\beta)=\underset{x}{E}\,[\bar{f}(x)>\alpha, \underline{f}(x)<\beta]\,. \end{cases}$$

Consequently, $\alpha>\beta$ implies mes $\big(E(\alpha,\beta)\big)=0$, and since this is true for any pair of real numbers α and β with $\alpha>\beta$, we have $\bar{f}(x)=\underline{f}(x)$ almost everywhere; that is,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}f(T^ix)=f_1(x)$$

exists almost everywhere.

Theorem 2. For any real number α we have

$$(2)\quad \begin{cases} \alpha \text{ mes}\big(E^*(\alpha)\big) \leqq \displaystyle\int_{E^*(\alpha)} f(x)\,dx\,, \quad \alpha \text{ mes}\big(E_*(\alpha)\big) \geqq \displaystyle\int_{E_*(\alpha)} f(x)\,dx\,, \\ \text{where}\quad E^*(\alpha)=\underset{x}{E}\,[f^*(x)>\alpha] \text{ and } E_*(\alpha)=\underset{x}{E}\,[f_*(x)<\alpha]\,. \end{cases}$$

Theorem 1 is the *Ergodic Theorem of Birkhoff* in its form given by A. Kolmogoroff.[1] Theorem 2 is new. We shall call Theorem 2

1) A. Kolmogoroff: Ein vereinfachter Beweis des Birkhoff-Khintchinschen Ergodensatzes, Recueil Math., **44** (1937), 366-368. See also E. Hopf: Ergodentheorie, Ergebnisse der Math., Heft **5** (1937).