52. On some Fundamental Theorems in the Theory of Operators in Hilbert Space.

By Kunihiko KODAIRA.
Mathematical Institute, Tokyo Imperial University.
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The purpose of this note is to point out that some fundamental theorems in the theory of operators in Hilbert space, viz.

I. the possibility of the canonical decomposition of closed linear operators with an everywhere dense domain;[1]

II. the possibility of the integral representation of normal and especially self adjoint operators[2]

are easily deducible from a certain lemma contained in the proofs of the lemmas 9. 1. 2, 9. 1. 3, 9. 1. 4 in R.[1]

We state this lemma in § 1, and then prove these fundamental theorems in the following two paragraphs. The sole knowledge presupposed to our demonstrations is that of operational calculus for bounded operators exposed for example in E,[2] Anhang III. Our proofs need not be altered, if we have to consider the generalized complex Euclidean space instead of the Hilbert space, as we make no use of the separability or of the infinite dimensionality of the space. §§ 1, 2 hold good also for the generalized real Euclidean space, so that our proof for I has somewhat larger validity than von Neumann's given in A.[3]

1. *Lemma. Let* $\mathfrak{A}$ *be a linear, everywhere dense set in Hilbert space (or in generalized Euclidean space)* $\mathfrak{H}:[\mathfrak{A}]=\mathfrak{H}$, *and* $Q(f, g)$ *be a complex-valued (or real-valued, if one has to consider the real Euclidean space) function of* $f, g \in \mathfrak{A}$, *having the properties of inner product in* $\mathfrak{A}$. *We suppose that* $\mathfrak{A}$ *is Q-complete, i. e. complete with respect to the metric determined by* $Q(f, f)$. *If* $Q(f, f) \geqq (f, f)$ *for all* $f \in \mathfrak{A}$, *then there exists a unique operator* B *in* $\mathfrak{H}$ *mapping* $\mathfrak{H}$ *in* $\mathfrak{A}$ *so that*

$$(1) \qquad Q(Bf, g)=(f, g) \quad \text{for} \quad f \in \mathfrak{H}, \quad g \in \mathfrak{A}.$$

B has the following properties:

1) *B is a bounded Hermitian operator and* $0<(Bf, f) \leqq (f, f)$ *if* $f \neq 0$; *so that we can form operators as* $\sqrt{B}$, $\sqrt{1-B}$ *in the sense of F. Riesz.*

2) $\mathfrak{A}=\text{Range}\ \sqrt{B}$

3) $Q(\sqrt{B}f, \sqrt{B}g)=(f, g)$ *for all* $f, g \in \mathfrak{H}$.

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1) J. v. Neumann: Über adjungierte Funktionaloperatoren, Ann. of Math. **33** (quoted as A). See also F. J. Murray and J. v. Neumann: On rings of operators, Ann. of Math. **37** (quoted as R.) especially p. 141–142. As to the notation and terminology we follow the usage in R.

2) J. v. Neumann: Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren, Math. Ann. **102** (quoted as E.) and Zur Algebra der Funktionaloperatoren, ibid.

3) We wish to remark, by the way, that we could verify that all results of R. hold with slight modifications for complex or real generalized Euclidean space. We reserve it for later publications.