21. An Abstract Integral, II.

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Introduction. The object of this paper is to define Riemann and Lebesgue integrals of functions whose values belong to a very general space. For this purpose, we will define new definitions of Riemann and Lebesgue integrals of real-valued functions of a real variable. More generally, the definitions of Riemann-Stieltjes and Lebesgue-Stieltjes integrals are given in §1 and §2, which are free from the notions of partition, least upper bound and greatest lower bound. These definitions make us easy to generalize these two integrals into abstract spaces. This is given in §3 and §4. The properties and applications of these integrals are left to the next occasion.

1. Let $f(x)$ and $z(x)$ be real-valued functions defined in the interval (a, b) and $z(x)$ be non-negative, montone and $z(b)-z(a)=1$. The sequence $(x_{n\nu})(n=1, 2, 3, \dots ; \nu=1, 2, \dots, n)$ is said to satisfy the $(C, 1)$ condition, provided that:

1.1.° $a \leqq x_{n\nu} \leqq b$.

1.2.° For any subinterval (α, β) of (a, b) such that $z(x)$ is continuous at $x=\alpha$ and $x=\beta$,

$$\lim_{n\to\infty} (M_n/n) = z(\beta) - z(\alpha) ,$$

where M_n denotes the number of $x_{n\nu}$ in (α, β) for fixed n.

If the limit

$$\lim_{n\to\infty} \frac{1}{n}\sum_{\nu=1}^{n} f(x_{n\nu}) \tag{1}$$

exists for any $(x_{n\nu})$ satisfying the $(C, 1)$ condition and the limiting value is independent of the choice of $(x_{n\nu})$, then we denote it by

$$(\mathfrak{R})\int_a^b f(x)dz(x)$$

and we say that $f(x)$ is $(\mathfrak{R})$-integrable.

If $f(x)$ is continuous, then it is easy to verify that $f(x)$ is $(\mathfrak{R})$-integrable. It is known that, if $f(x)$ is continuous, (1) is equal to the Riemann-Stieltjes integral (or simply (RS)-integral) of $f(x)$ by the determinate function $z(x)$.[1] Further we can prove that $(\mathfrak{R})$-integral is equivalent to the (RS)-integral.[2]

2. Let $f(x)$ be a real-valued measurable function in (a, b) and $z(e)$

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1) J. Schoenberg, Math. Zeits., **28** (1928).

2) Cf. I. J. Ridder, Prace Mat.-Fys., 1936.