64. *Note on Uni-serial and Generalized Uni-serial Rings.*

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The purpose of this short note is to make some supplementary remarks on my papers "On Frobeniusean algebras" I[1] and II.[2] The remarks are about uni-serial[3] and generalized uni-serial rings[4] as well as about principal two-sided ideals.

Let A be a ring satisfying the minimum and the maximum condition for left and right ideals.[5] As an application of our study of Frobeniusean rings, we showed (F. I, Theorem 10; F. II, Theorem 16):

Theorem 1. If every two-sided ideal $\mathfrak{z}$ *in* A *is expressible as* $\mathfrak{z}=Ac=cA$ $(c\in A)$ *then every residue class ring of* A*, including* A *itself, is Frobeniusean,*[6] *and conversely.*

On the other hand, K. Asano proved in his paper "Verallgemeinerte Abelsche Gruppe mit hyperkomplexem Operatorenring und ihre Anwendungen":[7]

Theorem 2. If every two-sided ideal $\mathfrak{z}$ *in* A *is expressible as* $\mathfrak{z}=Ac=dA$ $(c, d\in A)$ *then* A *is uni-serial, and conversely.*[8]

Notwithstanding their apparent differences these two theorems express, as the writer realized later, one and the same fact.[9] Indeed we have, first, the following lemma, which is perhaps of some interest for itself:

Lemma 1. Let A *possess a unit element. If a two-sided ideal* $\mathfrak{z}$ *of* A *is expressible as* $\mathfrak{z}=Ac=dA$*, then* $\mathfrak{z}=cA=Ad$ *too.*

Proof. Denote the composition length of a left A-module $\mathfrak{m}$ by $[\mathfrak{m}]_l$. From the mapping $a\rightarrow ac$ we see readily that[10] $\mathfrak{z}=Ac$ is isomorphic to $A/l(c)=A/l(cA)$ whence $[\mathfrak{z}]_l=[A/l(cA)]_l$. But $cA\leqq\mathfrak{z}$ and

1) Ann. Math. **40** (1939) — referred to as F. I.

2) Forthcoming in Ann. Math. — referred to as F. II.

3) Einreihig. See G. Köthe, Verallgemeinerte Abelsche Gruppe mit hyperkomplexem Operatorring, Math. Zeitschr. **39** (1934). Cf. also F. I, § 7.

4) F. II, § 9. Cf. also F. I, § 2.

5) A may have an operator domain of the type described in F. II, § 4.

6) F. II, § 4. See also F. I, § 2.

7) Japanese Journ. Math. **15** (1939).

8) Not only that, every left or right ideal of a uni-serial ring is principal.
As for the connection between uni-serial rings and principal ideals cf. also the writer's note, A note on the elementary divisor theory in non-commutative domains, Bull. American Math. Soc. **44** (1938), and K. Asano, Nicht-kommutative Hauptidealringe, Act. sci. ind. (1938).

9) Incidentally, the remark adjoining the definition of generalized uni-serial rings (F. II, § 9) was rather redundant; See Lemma 1 below. But our Theorem 17 there (as well as F. I, Theorem 11) retains, of course, its original significance, since there are certainly generalized uni-serial rings which are not uni-serial. (For that Theorem 17 cf. the second half of the present note.) For instance, an algebra consisting of all matrices of a given degree $\geqq 2$ such that all the coefficients above the diagonal vanish is such.

10) We denote by $l(S)$, $S\leqq A$, the set of left annihilators of S in A.