63. *An Abstract Treatment of the Individual Ergodic Theorem.*

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§ 1. *Introduction.* The analytical interests concerning the ergodic theorems may be observed from two viewpoints. Firstly, the ergodic theorems give us means for determining the fixed points und linear operations. Secondly, they concern with the deduction of the stronger convergences of linear operations from the weaker ones. Some entensive literatures on the mean ergodic theorem are more or less guided by these viewpoints. Concerning the individual ergodic theorem, however, we have only small number of literatures. The dominated ergodic theorem due to N. Wiener and M. Fukamiya,[1] and the writer's extensions[2] of Birkhoff-Khintchine's ergodic theorem both constitute examples on the individual ergodic theorem. The purpose of the present note is to extend the idea developed in [I]. The continuity theorems (theorem 1 and its corollary) have interests of their own. Theorem 2 is an abstract form of the individual ergodic theorem. We may deduce from this the individual ergodic theorem for m-parameter abelian group of equi-measure transformations, and more generally that for general semi-group of linear operations (theorem 3). The results in [I] in a somewhat extended form are also deducible from theorem 3.

§ 2. *A continuity theorem and an abstract form of the individual ergodic theorem.* Under abstract (S) space $(A-S)$ we mean a linear space of type-F, which satisfies the following axioms (we donote by $x, y, \dots$ the elements of $(A-S)$, by $\|x\|_S, \|y\|_S, \dots$ their quasi-norms and by λ a real scalar):

(1) a semi-order relation $x > y$ is defined in $(A-S)$, relative to which $(A-S)$ is a linear lattice, viz.:

(1–1) corresponding to any two elements x, y there exist the least upper bound $\sup(x, y)$ and the greatest lower bound $\inf(x, y)$,

(1–2) translations $x \to x+y$ and homothetic expansions $x \to \lambda x$ with $\lambda > 0$ preserve the semi-ordering,

(1–3) $\sup(x, y)$ and $\inf(x, y)$ are both continuous in x and y in the topology defined by the norm $\| \ \|_S$.

(2) any sequence $\{x_n\}$ bounded from above (below) admits of the least upper bound $\sup_n x_n$ (the greatest lower bound $\inf_n x_n$).

(3) if we write $\operatorname*{Lim}_{n\to\infty} x_n = x$, in case $\overline{\operatorname*{Lim}_{n\to\infty}} x_n = \inf_n \sup_{m\ge n} x_m = \underline{\operatorname*{Lim}_{n\to\infty}} x_n = \sup_n \inf_{m\ge n} x_m$, then $\operatorname*{Lim}_{n\to\infty} x_n = x$ implies $\lim_{n\to\infty} \|x_n - x\|_S = 0$.

(4) $x \geqq y \geqq -x$ implies $\|x\|_S \geqq \|y\|_S$.

1) N. Wiener: Duke Math. J., **5** (1939), 1–18. M. Fukamiya: Tôhoku Math. J., **46** (1939), 150–153. Cf. also K. Yosida and S. Kakutani: Proc. **15** (1939), 165–168.

2) K. Yosida: Jap. J. of Math., **15** (1940), 31–36, to be cited as [I] below.