60. On One-parameter Groups of Operators.

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1. Theorem. Let E be a separable Banach space, and let $\{U_t\}$, $(-\infty < t < \infty)$ be a one-parameter group of operators on E to E such that: (1) $\| U_t \| = 1$, (2) $U_t U_s = U_{t+s}$ for any t, s, (3) $f(U_t x)$ is measurable in t for every $x \varepsilon E$ and for every $f \varepsilon \bar{E}$. Then there exist the operators R_z (resolvents) and A, which satisfy the following properties:

(1) R_z is defined for every complex number z, with $\mathcal{I}_m(z) \neq 0$,

(2) R_z is a bounded, linear operator on E to E, and $\| R_z \| \leqq \dfrac{1}{|\mathcal{I}_m(z)|}$,

(3) $(z-z') R_z R_{z'} = R_z - R_{z'}$, for every z, z' with $\mathcal{I}_m(z) \neq 0$, $\mathcal{I}_m(z') \neq 0$,

(4) $R_z x = 0$ implies $x = 0$, for any z;

(5) A is a closed linear oparator on E to E, whose domain $D(A)$ is dense in E, and

$$(A - zI) \cdot R_z = I, \qquad R_z(A - zI) = I \ \big(\text{in } D(A)\big),$$

(6) For any $x \varepsilon D(A)$, $\quad \lim_{t \to 0} \dfrac{U_t - 1}{t} \cdot x = A \cdot x.$

We will prove these results, following the method of M. H. Stone.[1] Recently similar facts were obtained by I. Gelfand.[2] But the method is completely different from ours.

2. Proof: Let $\psi(\tau; z)$ be defined by

$$\psi(\tau; z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\lambda - z} e^{-i\lambda\tau} d\lambda \qquad \big(\mathcal{I}_m(z) \neq 0\big)$$

$$= \begin{cases} 0 & \tau > 0 \\ ie^{-iz\tau} & \tau < 0 \end{cases} \big(\mathcal{I}_m(z) > 0\big), \quad = \begin{cases} -ie^{-iz\tau} & \tau > 0 \\ 0 & \tau < 0 \end{cases} \big(\mathcal{I}_m(z) < 0\big).$$

Then

(i) $\dfrac{1}{\lambda - z} = \displaystyle\int_{-\infty}^{\infty} \psi(\tau; z) e^{i\lambda\tau} d\tau$,

(ii) $(z - z') \displaystyle\int_{-\infty}^{\infty} \psi(\tau; z) \psi(\sigma - \tau; z') d\tau = \psi(\sigma; z) - \psi(\sigma; z')$,

(iii) $\overline{\psi(\tau; z)} = \psi(-\tau; \bar{z})$.

We define $F(f)$ by

$$F(f) = \int_{-\infty}^{\infty} \psi(\tau; z) f(U_\tau x) d\tau, \quad f \varepsilon \bar{E}, \quad x \varepsilon E \text{ and } \mathcal{I}_m(z) \neq 0.$$

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1) M. H. Stone, Linear Transformations in Hilbert Space, 1932, Chap. IV, V; Annals of Math., **33** (1932), pp. 643–648.
J. von Neumann, Annals of Math., **33** (1932), pp. 567–573.

2) Gelfand, C. R. U. R. S. S., **25** (1939).