79. Concircular Geometry II. Integrability Conditions of $\rho_{\mu\nu}=\phi g_{\mu\nu}$.

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(Comm. by S. KAKEYA, M.I.A., Oct. 12, 1940.)

In a previous paper entitled Concircular geometry I,[1] we have considered, in a Riemannian space, curves defined by

$$(0.1)\qquad \frac{\delta^3 u^\lambda}{\delta s^3}+\frac{\delta u^\lambda}{\delta s}g_{\mu\nu}\frac{\delta^2 u^\mu}{\delta s^2}\frac{\delta^2 u^\nu}{\delta s^2}=0\,,\qquad (\lambda,\mu,\nu,\dots=1,2,3,\dots,n)\,,$$

which may be regarded as a generalization of circles in ordinary euclidean space, and we have called them *geodesic circles*. If a conformal transformation

$$(0.2)\qquad \bar{g}_{\mu\nu}=\rho^2 g_{\mu\nu}$$

of the fundamental metric tensor $g_{\mu\nu}$ transforms any geodesic circle into a geodesic circle, then the functiou ρ must satisfy the following partial differential equations

$$(0.3)\qquad \rho_{\mu\nu}\equiv\rho_{\mu,\nu}-\rho_\lambda\left\{{}^{\lambda}_{\mu\nu}\right\}-\rho_\mu\rho_\nu+\frac{1}{2}g^{\alpha\beta}\rho_\alpha\rho_\beta g_{\mu\nu}=\phi g_{\mu\nu}\,,\qquad \left(\rho_\mu=\frac{\partial\log\rho}{\partial u^\mu}\right).$$

We have called such a conformal transformation a *concircular transformation.*

In the present Note, we shall consider the integrability conditions[2] of the partial differential equations (0.3).

§ 1. The function ρ satisfying the equations

$$(1.1)\qquad \rho_{\mu;\nu}-\rho_\mu\rho_\nu+\frac{1}{2}g^{\alpha\beta}\rho_\alpha\rho_\beta g_{\mu\nu}=\phi g_{\mu\nu}\,,$$

where the semi-colon denotes the covariant derivative, we have

$$(1.2)\qquad \rho_{\mu;\nu}-\rho_\mu\rho_\nu=\psi g_{\mu\nu}\,,$$

where

$$(1.3)\qquad \psi=\phi-\frac{1}{2}g^{\alpha\beta}\rho_\alpha\rho_\beta\,.$$

Consequently, putting

$$(1.4)\qquad \rho^\lambda=g^{\lambda\mu}\rho_\mu\,,$$

we obtain, from (1.2),

$$(1.5)\qquad \rho^\lambda{}_{;\nu}\rho^\nu=\rho^\lambda(\psi+\rho_\alpha\rho^\alpha)\,.$$

1) K. Yano, Concircular geometry I. Concircular transformations. Proc. **16** (1940), 195–200.

2) This problem was also studied by A. Fialkow, Conformal geodesics, Trans. Amer. Math. Soc. **45** (1939), 443–473.