117. On Linear Functions of Abelian Groups.

By Kôshichi TOYODA.
Harbin Technical College, Manchoukou.
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1. Let a set G of elements $a_i, b_i, c_i \dots\dots$, $(i=1, 2, \dots, n)$, satisfy the following axioms:

(1) There exists an operation in G which associates with each class of n elements $a_1, a_2, \dots, a_n$ of G an $(n+1)$-th element a_0 of G, i.e.,

$$(a_1, a_2, \dots, a_n) = a_0 .$$

(2) The operation satisfies the associative law

$$\big((a_1, a_2, \dots, a_n),\ (b_1, b_2, \dots, b_n),\ \dots\dots,\ (d_1, d_2, \dots, d_n)\big)$$
$$= \big((a_1, b_1, \dots, d_1),\ (a_2, b_2, \dots, d_2),\ \dots\dots,\ (a_n, b_n, \dots, d_n)\big) .$$

(3) There exists at least one unit element 0 such that

$$(0, 0, \dots, 0) = 0 .$$

(4) For any given elements a, b, each of the equations

$$(x, a, 0, \dots, 0) = b \quad \text{and} \quad (a, y, 0, \dots, 0) = b$$

has a unique solution with respect to the unknown x and y respectively.

We know[1] that the mean value of n real numbers $x_1, x_2, \dots, x_n$, say,

$$(x_1, x_2, \dots, x_n) = \frac{x_1 + x_2 + \dots + x_n}{n}$$

satisfies the above axioms (1), (2), (4), and, in place of (3), the axiom: "every element is unit element," and the symmetrical condition. We shall consider the converse problem which is answered as follows:

Theorem[2]. *The set G forms an abelian group with respect to the new operation which is defined by the equation*

$$x + y = (a, b, 0, \dots, 0) ,$$

assuming that $x = (a, 0, 0, \dots, 0)$ *and* $y = (0, b, 0, \dots, 0)$.

Moreover, the operation $(x_1, x_2, \dots, x_n)$ of G is expressed as a linear function of $x_1, x_2, \dots, x_n$ such that

$$(x_1, x_2, \dots, x_n) = A_1 x_1 + A_2 x_2 + \dots + A_n x_n ,$$
$$A_i A_k = A_k A_i , \qquad (i, k = 1, 2, \dots, n) ,$$

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1) This result is due to the remark of Mr. M. Takasaki.

2) K. Toyoda, On Axioms of Mean Transformations and Automorphic Transformations of Abelian Groups, Tôhoku Math. Journal, **47** (1940), pp., 239–251.

K. Toyoda, On Affine Geometry of Abelian Groups, Proc. **16** (1940), 161–164.