115. Lattice Theoretic Foundation of Circle Geometry.

By Shin-ichi IZUMI.

Mathematical Institute, Tohoku Imperial University, Sendai.

(Comm. by M. FUJIWARA, M.I.A., Dec. 12, 1940.)

By G. Birkhoff and K. Menger the lattice theoretic foundation of projective geometry, affine geometry and non-euclidean geometry was established, from which continuous geometry of von Neumann was obtained as a generalization. On the other hand, the axiomatic foundation of the circle geometry was already established[1], but axioms of projective, affine and non-euclidean geometries are not sufficient for circle geometry. The object of this paper is therefore to give the system of axioms of circle geometry from the lattice theoretic standpoint.

1. Let L be a system of elements $a, b, c, \dots$ which satisfies the following axioms.

Axiom 1. L is a partially ordered system with 0 and I, that is, the relation "$\leqq$" is defined and

1.1. $a \leqq a$.

1.2. *$a \leqq b$ and $b \leqq c$ imply $a \leqq c$.*

1.3. *there exist 0 and I such that $0 \leqq a \leqq I$ for all a in L.*

Axiom 2. L is a semi-lattice, that is, the relations join "$\cup$" and meet "$\cap$" are defined and

2.1. *for every a and b in L, there exists $c = a \cup b$, such that (i) $a \leqq c$, $b \leqq c$ and (ii) $a \leqq d$, $b \leqq d$ imply $c \leqq d$.*

2.2. *if $e = a \cap b$ exists, then (i) $e \leqq a$, $e \leqq b$ and (ii) $f \leqq a$, $f \leqq b$ imply $f \leqq e$.*

Axiom 3. L is finite dimensional, that is, for every a in L there corresponds a finite positive integer $d(a)$, called dimension function such that

3.1. $d(0) = -1$, $d(I) > 3$.

3.2. $1 + max\big(d(a), d(b)\big) \leqq d(a \cup b) \leqq 1 + d(a) + d(b)$ *if* $a \cup b \neq a, b$.

Axiom 4. L is modular in a restricted sense, that is,

4.1. *if (i) $a \leqq c$, (ii) $(a \cup b) \cap c$ and $b \cap c$ exist and (iii) c covers a or b covers $b \cap c$, then*

$$(a \cup b) \cap c = a \cup (b \cap c).$$

2. We will show that circle geometry satisfies these axioms. Let us consider a system $\mathfrak{L}$ of elements of the following kinds: point, a pair of points, circle, sphere, ..., whose dimensions are $0, 1, 2, 3, 4, \dots$, respectively.

Taking $\leqq$ as incidence relation, $\mathfrak{L}$ satisfies Axiom 1. Concerning Axiom 2, join $\cup$ is interpreted as follows: join of two points is a pair of points, join of a point and a pair of points is a circle determined

---

1) This point has been communicated by Prof. T. Kubota. This problem was suggested to the author by Mr. S. Kyo whom he expresses his hearty thanks.