PAPERS COMMUNICATED

1. An Abstract Integral, IV.

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In the integral defined in this paper, the concept of "function" is dropped and that of "partition" is abstracted. The "abstract partition" contains as particular cases closure operations and transformations appearing in the ergodic theory. And then thus developed integration theory contain closure topology and ergodic theory.

1. Let $\mathbb{L}$ be a vector lattice,[1] that is, $\mathbb{L}$ is a linear space and the relation "$f \geqq 0$" is defined for some $f \in \mathbb{L}$ such as ($f, g, \dots$ are elements of $\mathbb{L}$ and $\lambda, \mu, \dots$ are real numbers)

($\mathbb{L}$, 1) if $f \geqq 0$ and $\lambda \geqq 0$, then $\lambda f \geqq 0$,

($\mathbb{L}$, 2) if $f \geqq 0$ and $g \geqq 0$, then $f+g \geqq 0$,

and further, if $f \geqq g$ is defined to mean $f-g \geqq 0$, then

($\mathbb{L}$, 3) the relation "$f \geqq g$" defines a lattice.

Let $\mathbb{T}$ be a certain space and $\{T^\alpha\}$ be a set of transformations from $\mathbb{L}$ to $\mathbb{T}$, such that

(T, 1) $\{T^\alpha\}$ is the Moore-Smith set or index set $\{\alpha\}$ has the Moore-Smith property, that is for any two α and β in $\{\alpha\}$, there exists a γ such that $\gamma \geqq \alpha$ and $\gamma \geqq \beta$ (or $\gamma \leqq \alpha$ and $\gamma \leqq \beta$), where "$\leqq$" is a transitive asymmetric relation. Further axioms for $\mathbb{T}$ and $\{T^\alpha\}$ will be added later.

Example 1. (L) (the space of integrable functions in $(0, 1)$) satisfies the axioms of $\mathbb{L}$. If we take $\mathbb{T} = R_1$ (set of all real numbers) and T^α as the operation taking Lebesgue sum, then xT^α becomes the Lebesgue sum of function x in (L). If the limit $\lim\limits_{\alpha} xT^\alpha$ is defined in ordinary manner, then it becomes the Lebesgue integral of x.

Starting from (M) (set of all real measurable functions) which satisfies also the axioms of $\mathbb{L}$, we get (L) as the set of all elements x such that $\lim\limits_{\alpha} xT^\alpha$ exists.

Example 2. Let $\mathbb{L} = \mathbb{T}$ and τ be a certain operator from $\mathbb{L}$ onto itself. Let $\tau^1 = \tau$, $\tau^n = \tau(\tau^{n-1})$ $(n > 1)$ and put $fT^n = \{f\tau + \cdots + f\tau^n\}/n$. We define $\lim\limits_{n} xT^n$. If the limit exists for an f and is equal to the "integral" of x, then such f is called ergodic or ergodic element.

Example 3. Let E be the set of all characteristic functions and their linear combinations on a certain topological space. E satisfies the axioms of $\mathbb{L}$. We take $\mathbb{T}$ as E itself and T^α as a closure operation. That is, for a characteristic function e in E, eT means closure in the ordinary sense and for non-characteristic function $f = \sum\limits_{i=1}^{n} \alpha_i e_i$ (e_i being

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1) We use the terminologies in Birkhoff, Lattice Theory.