PAPERS COMMUNICATED

13. *On Regularly Convex Sets.*

By Kôsaku YOSIDA and Masanori FUKAMIYA.
Mathematical Institute, Osaka Imperial University.

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§ 1. *Introduction.* We denote by E a Banach space and by $\bar{E}$ its conjugate space. A set $X \leqq E$ is called *convex* if $x, y \in E$ implies $ax+(1-a)y \in E$ for all $a\,(1 \geqq a \geqq 0)$. According to M. Krein and V. Smulian[1], a set $F \leqq \bar{E}$ is called *regularly convex* if for every $g \bar{\in} F$ $(g \in \bar{E})$ there exists $x_0 \in E$ such that $\sup_{f \in F} f(x_0) < g(x_0)$. It is easy to see that only convex sets in $\bar{E}$ may be regularly convex. Moreover we may prove

Theorem 1. A convex set $F \leqq \bar{E}$ is regularly convex if and only if F is closed in the weak topology of $\bar{E}$ as functionals.

Hereby, for any $f_0 \in \bar{E}$, its weak neighbourhood $U(f_0, x_1, x_2, \dots, x_n, \varepsilon)$ is defined as the totality of $f \in \bar{E}$ such that $\sup_{1 \leqq i \leqq n} |f(x_i) - f_0(x_i)| < \varepsilon$, where $\{x_i\}_{i=1,2,\dots,n}$ is an arbitrary system of points $\in E$ and ε is an arbitrary positive number[2].

The purpose of the present note is to show that there exists a kind of duality between (strongly) closed convex sets $\leqq E$ and regularly convex sets $\leqq \bar{E}$. By this duality we may give, to almost all the theorems in Chapter 1 of [$K-S$], geometrical interpretations and new proofs. We then give a proof to Krein-Milman's[3]

Theorem 2. If a (strongly) bounded set $F \leqq \bar{E}$ is regularly convex, then F has extreme points f_0, viz. points f_0 such that $f_0 \neq \frac{1}{2}(g+h)$ for any two $g, h \in F$, $g \neq f_0$, $h \neq f_0$.

If E is separable, the above duality shows that Theorem 2 is an immediate corollary of a theorem due to S. Mazur[4]. The proof for non-separable E is also given by tranfinite induction. This was obtained by one of us (Fukamiya): Zenkoku Shijyô-Sugaku Danwakai, 207 (Japanese). After the present note is completed, we received a letter from S. Kakutani, now in Princeton, and we knew that F. Bohnenblust

1) Ann. of Math., **41** (1940), 556–583, to be referred as [$K-S$].

2) The importance of weak topologies in the theory of Banach spaces was especially stressed on by S. Kakutani with much success: Proc. **15** (1939), 169–173 and **16** (1940), 63–67. We omit the easy proof of Theorem 1, since it is similar to his proof of the equivalence between the transfinite closure and the regular closure of linear subspaces $\leqq \bar{E}$. The equivalence of the two notions: transfinitely closed convex sets $\leqq \bar{E}$ and regularly convex sets $\leqq \bar{E}$, was proved in [$K-S$], 569.

3) The vol. 9 of the Stud. Math. is not yet arrived at our institute. We knew their result from M. and S. Krein's paper: C. R. URSS, **27**, 5 (1940), 427–430.

4) Stud. Math., **4** (1933), 70–84.