PAPERS COMMUNICATED

50. On Axioms of Linear Functions.

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1. Let a set G of elements $a, b, c, \dots$, satisfy the following axioms:

(1) There exists an operation in G which associates with each pair a, b of G an element c of G, i. e., $a\cdot b=c$.

(2) The operation satisfies the associative law:

$$(a\cdot b)\cdot(c\cdot d)=(a\cdot c)\cdot(b\cdot d).$$

(3) If a, b, c are any given elements, each of the equations $a\cdot x=c$ and $x\cdot b=c$ is uniquely soluble in G for x.

As an example, we show that a real linear function of two real variables x, y, i. e.,

$$x\cdot y=\lambda x+\mu y+\nu$$

satisfies the above axioms (1), (2), (3). Conversely, we shall prove

Theorem 1[1]. *The set G forms an abelian group with respect to the new operation $x+y=z$ which is defined by the equation*

$$a\cdot s+r\cdot b=a\cdot b,$$

where r and s denote two fixed elements in G.

Furthermore, the operation $x\cdot y$ of G is expressed as a linear function of x, y with respect to the new operation such that

$$x\cdot y=Ax+By+c,$$

where A and B denote the automorphisms of G and are mutually permutable, that is, $AB=BA$, and c is a fixed element in G.

Next, let us consider a set G^* of elements $a, b, c, \dots$, which satisfies the axioms (1), (2) and the axiom

(3*) There exists at least one unit element 0 in G^*, i. e., $0\cdot 0=0$ and, if a is any given element, each of the equations $x\cdot 0=a$ and $0\cdot x=a$ has at least one solution in G^* for x.

As examples, we show that the sum (or product) of two sets a, b of points, i. e.,

$$a\cdot b=a\dot{+}b\dot{+}0$$

and a linear differential expression of two real functions $x(t)$, $y(t)$ of a real variable t, i, e.,

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