13. An Abstract Integral, VII.

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Introduction. This is the preliminary report of analysis of functions with range in a complete vector lattice. This subject was firstly studied by S. Bochner[1]. § 1 contains the definitions and theorems of measurable functions. The definition of the measurability is that of S. Bochner. § 2 is the integration theory. Integral is defined by the idea of McNeille[2]. § 3 contains some remarks on integrals. Some related integrals are introduced and a modified integral is shown to coincide with the Bochner integral[3] when the range is the Banach lattice. § 4 is the Fourier series theory. Here the Bessel inequality is proved. This is not true for the Bochner integral[3] with range in the Banach space. This point is a reason why we develop the analysis of functions with range in a lattice in stead of a Banach space. § 5 is a generalization of § 1 and § 2. The content of § 5 shows that the theory of integral and that of measure can be placed under a general theory. In the ordinary theory one of those theories is derived from the other[4].

§ **1.** *Measurable functions*[5].

[1.1] I is a fixed finite interval in an Euclidean space.

[1.2] V is a fixed σ-complete vector lattice.

We will consider functions with domain I and with range in V and will denote them by $f(x)$ and $g(x)$, etc. Such functions are supposed to be defined uniquely in a full set of I and need not be defined in the complementary null set.

[1.3] $f(x)$ is called a simple function if there are an integer n, a set of real numbers $(a_1, a_2, \dots, a_n)$ and a set of disjoint measurable sets $(E_1, E_2, \dots, E_n)$ such that

$$I=\sum_{k=1}^{n} E_k, \quad f(x)=a_k \text{ in } E_k \ (k=1, 2, \dots, n).$$

[1.4] $f(x)$ is called to be measurable if there is a sequence of simple functions $f_n(x) (n=1, 2, \dots)$ such that $f_n(x)$ tends to $f(x)$ relative uniformly almost everywhere, that is, there are sequences of functions $\lambda_n(x)$, $g_n(x) (n=1, 2, \dots)$ such that $\lambda_n(x)$ tends to zero monotonously (by the order topology) almost everywhere as $n \to \infty$ and $|f_n(x)-f(x)| \leqq \lambda_n(x) g(x)$ almost everywhere. We write $f_n(x) \to f(x)$ (r. u.) a. e. or $f(x) = $ (r. u)-$\lim f_n(x)$, a. e.

If $f(x)$ is measurable, then we write $f(x) \in M$.

1) S. Bochner, Proc. Nat. Academy, (1939).
2) McNeille, ibidem (1941).
3) S. Bochner, Fund. Math., **20** (1930).
4) cf. S. Izumi, An Abstract integral IV, Proc. Imp. Acad. of Japan, (1941).
5) [ ], ( ) and { } denote definition, theorem and axiom respectively.