37. On Green's Lemma.

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1. We will prove the well known Green's lemma in the following generalized form.

Theorem. Let D be a domain on the $z=x+iy$-plane, bounded by a rectifiable curve Γ and $A(z)=A(x,y)$, $B(z)=B(x,y)$ be continuous and bounded functions of z inside D, which satisfy the following conditions:

(i) *$\lim A(z)$, $\lim B(z)$ exist almost everywhere on Γ, when z tends to Γ non-tangentially.*

(ii) *$A(x, y_0)$ is an absolutely continuous function of x on the part of the line $y=y_0$, which lies in D, for almost all values of y_0 and $B(x_0, y)$ is an absolutely continuous function of y on the part of the line $x=x_0$, which lies in D, for almost all values of x.*

(iii) $\iint_D \left(\left| \frac{\partial A}{\partial x} \right| + \left| \frac{\partial B}{\partial y} \right| \right) dx\,dy$ *is finite.*

Then

$$\iint_D \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} \right) dx\,dy = \int_\Gamma \left(A(z)\frac{dy}{ds} - B(z)\frac{dx}{ds} \right) ds,$$

where ds is the arc element on Γ and the line integral around Γ is taken in the positive sense.

The extension of Green's lemma for a domain D, bounded by a rectifiable curve was first proved by W. Gross[1] under the condition that $A(z)$, $B(z)$ are continuous in the closed domain $D+\Gamma$ and $\frac{\partial A}{\partial x}$, $\frac{\partial B}{\partial y}$ are continuous in D. Recently W. T. Reid[2] proved another extension under the condition that $A(z)$, $B(z)$ are continuous in the closed domain $D+\Gamma$ and the conditions (ii) and (iii) of our theorem.

We remark that since $A(x,y)$ is continuous, the Dini's derivatives:

$$\bar{A}_x^+(x,y) = \overline{\lim_{h\to+0}} \frac{A(x+h,y)-A(x,y)}{h},$$

$$\underline{A}_x^+(x,y) = \underline{\lim_{h\to+0}} \frac{A(x+h,y)-A(x,y)}{h}$$

are B-measurable functions of (x,y)[3], so that the set E in which $\bar{A}_x^+(x,y)=\underline{A}_x^+(x,y)$ is measurable. By the condition (ii), $\frac{\partial A}{\partial x}$ exists al-

1) W. Gross: Das isoperimetrische Problem bei Doppelintegralen. Monathefte f. Math. u. Phys. **27** (1927).

2) W. T. Reid: Green's lemma and related results. Amer. Journ. Math. **17** (1941).

3) Saks: Theory of the integral. p. 170.