46. On an Extension of Löwner's Theorem.

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We will prove the following extension of Löwner's theorem.

Theorem. *Let $w=f(z)$ be regular and $|f(z)|<1$ in $|z|<1$, $f(0)=0$ and $\lim_{r\to 1} f(re^{i\theta})=e^{i\psi}$ exists, when θ belongs to a set E and the ψ-set on $|w|=1$ be denoted by E^*. Then E and E^* are measurable and*

$$mE \leqq mE^* . \tag{1}$$

If $0<mE<2\pi$, then $mE<mE^*$.

Mr. Y. Kawakami[1] proved (1) under the condition that $f(z)$ is schlicht in $|z|<1$ and Messrs. S. Kametani and T. Ugaheri[2] proved that $m_iE \leqq m_eE^*$, where m_iE and m_eE denote the inner and outer measure of E.

Proof. Since $f(re^{i\theta})$ $(0<r<1)$ is continuous in $0\leqq\theta\leqq 2\pi$, by H. Hahn's theorem[3], the set e, where $\lim_{r\to 1} f(re^{i\theta})=\rho(\theta)e^{i\psi(\theta)}$ exists, is $F_{\sigma\delta}$, so that $\rho(\theta)$ and $\psi(\theta)$ are Borel functions defined on a Borel set e and hence the sub-set E of e, where $\rho(\theta)=1$, is a Borel set. Consider on the (θ, ψ)-plane a set M, whose points are $\big(\theta, \psi(\theta)\big)$, where $\theta\in E$. We will prove that M is a Borel set on the (θ, ψ)-plane.

Let $0=a_0<a_1<\cdots<a_{n-1}<a_n=2\pi$, $a_k-a_{k-1}=\dfrac{1}{n}$ $(1\leqq k\leqq n)$ and

$$E_k=E\big(a_{k-1}\leqq\psi(\theta)\leqq a_k\big) ,$$

$$\underline{M}_k=\text{the set of points } (\theta, \psi), \text{ where } \theta\in E_k ,\quad 0\leqq\psi<a_{k-1} ,$$

$$\underline{M}(n)=\sum_{k=1}^{n}\underline{M}_k ,$$

and

$$\overline{M}_k=\text{the set of points } (\theta, \psi), \text{ where } \theta\in E_k ,\quad 0\leqq\psi\leqq a_k ,$$

$$\overline{M}(n)=\sum_{k=1}^{n}\overline{M}_k .$$

Then for $n\to\infty$, $\underline{M}(n)\to\underline{M}$, $\overline{M}(n)\to\overline{M}$, so that $M=\overline{M}-\underline{M}$. Since $\overline{M}(n)$, $\underline{M}(n)$ are Borel sets, $\overline{M}$ and $\underline{M}$ and hence M is a Borel set. E^*, being the projection of M on the ψ-axis, is an analytic set, so that is measurable.

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1) Y. Kawakami; On an extension of Löwner's lemma. Japan. Jour. of Math. **17** (1941).

2) S. Kametani and T. Ugaheri: A remark on Kawakami's extension of Löwner's lemma. Proc. **18** (1942), 14.

3) Hausdorf. Mengenlehre, p. 271.