68. *Note on Banach Spaces (IV): On a Decomposition of Additive Set Functions.*

By Masahiro NAKAMURA and Gen-ichirô SUNOUCHI.
Mathematical Institute, Tohoku Imperial University, Sendai.
(Comm. by M. FUJIWARA, M.I.A., July 13, 1942.)

This paper is devoted to prove an abstract decomposition theorem from which flow three types of decompositions. The first is the decomposition theorem concerning cardinal number which is due to R. S. Phillips[1], the second concerns with category, and the last concerns with Lebesgue measure which is due to H. Hahn[2]. The second type seems to be new. In the proof of the theorem of the third type, Pettis' theorem is used. Since the complete proof is not yet published, we give it in the last section.

Throughout this paper, we denote by L an abstract Boolean algebra and by $x(e)$ a completely additive function from L to a Banach space. And finally we suppose I is a σ-ideal in L. Obviously, in the real valued case, our proof is also applicable to bounded, finitely additive set functions.

1. Let $\{e_a\}$ be a set in L such that $\{e_a\}$ is a disjoint system in I, and $x(e_a) \neq 0$ for all e_a[3]. Such $\{e_a\}$ form evidently a system Γ with finite character, thus by the use of Zorn's lemma, Γ contains a maximal collection $\{e_a^1\}$.

Since $\{e_a^1\}$ is at most countable, $a = \bigvee_a e_a^1$ exists and belongs to I. If we put

$$x'(e) = x(a \frown e) \quad \text{and} \quad x''(e) = x(a' \frown e),$$

then $x = x' + x''$. We will now prove the unicity of decomposition. Let $\{e_a^2\}$ be another maximal collection and put $b = \bigvee_a e_a^2$. By the identity $(a \frown e) \smile \{(b-a) \frown e\} = (b \frown e) \smile \{(a-b) \frown e\}$ and $x\{(b-a) \frown e\} = x\{(a-b) \frown e\} = 0$ we have $x(a \frown e) = x(b \frown e)$ for all e.

Summing up above results we get

Theorem 1. For any σ-ideal I in L. We can find an $a \in I$ such that the decomposition

$$x(e) = x(a \frown e) + x(a' \frown e) \tag{1}$$

is unique and the second part vanishes for all elements of the ideal.

2. We will now give applications of Theorem 1.

Let L be a Borel field of subsets of a space, and I be the family of all sets whose cardinal numbers do not exceed an infinite $\aleph$. Then Theorem 1 reads as

---

1) R. S. Phillips, Bull. of A. M. S., **46** (1940), 274-277. Idea of our proof is essentially due to him.

2) H. Hahn, *Theorie der reellen Fnnktionen*, 1. Band, Berlin 1921, p. 422.

3) In the proof of Phillips, the last restriction is dropped.