89. Concircular Geometry V. Einstein Spaces.

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§0. In four recent papers[1], we have defined the concircular transformations in the Riemannian spaces and studied the so-called concircular geometry in these spaces.

The concircular transformation is defined as a conformal transformation

$$\bar{g}_{\mu\nu}=\rho^2 g_{\mu\nu} \tag{0.1}$$

of Riemannian metric which satisfies the relation

$$\rho_{\mu\nu}=\phi g_{\mu\nu}\,, \tag{0.2}$$

where

$$\rho_{\mu\nu}=\rho_{\mu;\,\nu}-\rho_\mu\rho_\nu+\frac{1}{2}g^{\alpha\beta}\rho_\alpha\rho_\beta g_{\mu\nu}\,,\qquad (\rho_\mu=\partial\log\rho/\partial u^\mu) \tag{0.3}$$

and the semi-colon means the covariant derivative with respect to the Christoffel symbols $\{{}^{\lambda}_{\mu\nu}\}$ formed with $g_{\mu\nu}$, ϕ being a certain function of the coordinates u^λ.

The concircular transformation (0.1) of the metric keeps unchanged the differential equations

$$\frac{\delta^3 u^\lambda}{\delta s^3}+\frac{du^\lambda}{ds}\,g_{\mu\nu}\,\frac{\delta^2 u^\mu}{\delta s^2}\,\frac{\delta^2 u^\nu}{\delta s^2}=0 \tag{0.4}$$

of the geodesic circle where $\delta/\delta s$ denotes the covariant differentiation along the curve, s being the arc length of the curve.

If a Riemannian space V_n admits the concircular transformation (0.1), it will be readily proved that the tensor

$$Z^{\lambda}_{\mu\nu\omega}=R^{\lambda}_{\mu\nu\omega}-\frac{R}{n(n-1)}\,(g_{\mu\nu}\delta^{\lambda}_{\omega}-g_{\mu\omega}\delta^{\lambda}_{\nu})\,, \tag{0.5}$$

and consequently the contracted tensor

$$Z_{\mu\nu}=Z^{\lambda}_{\mu\nu\lambda}=R_{\mu\nu}-\frac{R}{n}\,g_{\mu\nu} \tag{0.6}$$

are invariant under these concircular transformations, where

$$R^{\lambda}_{\mu\nu\omega}=\{{}^{\lambda}_{\mu\nu}\}_{,\,\omega}-\{{}^{\lambda}_{\mu\omega}\}_{,\,\nu}+\{{}^{\alpha}_{\mu\nu}\}\{{}^{\lambda}_{\alpha\omega}\}-\{{}^{\alpha}_{\mu\omega}\}\{{}^{\lambda}_{\alpha\nu}\} \tag{0.7}$$

and

$$R_{\mu\nu}=R^{\lambda}_{\mu\nu\lambda}\,,\qquad R=g^{\mu\nu}R_{\mu\nu}\,, \tag{0.8}$$

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1) K. Yano: Concircular geometry I. Concircular transformations, Proc. **16** (1940), 195–200, II. Integrability conditions of $\rho_{\mu\nu}=\phi g_{\mu\nu}$, ibid. pp. 354–360, III. Theory of curves, ibid. pp. 442–445, IV. Theory of subspaces, ibid. pp. 505–511. These papers will be cited as C. G. I, II, III and IV.