87. *On the Function whose Imaginary Part on the Unit Circle Changes its Sign only Twice.*

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I. We are going to consider the function

$$f(z)=\sum_{n=1}^{\infty} c_n z^n = c_1 z + c_2 z^2 + \cdots \tag{1}$$

which is *regular within the unit circle and is continuous*, for simplicity, *to the boundary*. Putting

$$z=re^{i\theta}, \qquad f(z)=u(r,\theta)+iv(r,\theta) \tag{2}$$

we confine ourselves to the function which *satisfies one of the following two conditions*:

$$\left.\begin{aligned} v(1,\theta)=v(\theta) &\geqq 0 \quad \text{for} \quad \sigma_1 \leqq \theta \leqq \sigma_2 \\ &\leqq 0 \quad \text{for} \quad 0 \leqq \theta \leqq \sigma_1 \quad \text{and} \quad \sigma_2 \leqq \theta \leqq 2\pi \end{aligned}\right\} \tag{3}$$

or

$$\left.\begin{aligned} v(\theta) &\leqq 0 \quad \text{for} \quad \sigma_1 \leqq \theta \leqq \sigma_2 \\ &\geqq 0 \quad \text{for} \quad 0 \leqq \theta \leqq \sigma_1 \quad \text{and} \quad \sigma_2 \leqq \theta \leqq 2\pi \end{aligned}\right\} \tag{4}$$

namely the imaginary part of $f(z)$ on the unit circle $|z|=1$ may change its sign only at two points $e^{i\sigma_1}$ and $e^{i\sigma_2}$. $(0 \leqq \sigma_1 < \sigma_2 \leqq 2\pi)$.

It is easily to be seen that the function

$$g(z)=e^{-i\frac{\sigma_1+\sigma_2}{2}} \times \frac{(e^{i\sigma_1}-z)(e^{i\sigma_2}-z)}{z} \tag{5}$$

becomes positive on the unit circle for $\sigma_1 < \theta < \sigma_2$ and negative for the remaining arc. Hence the function

$$\begin{aligned} F(z)=\varepsilon f(z) g(z) &= \sum_{n=0}^{\infty} C_n z^n = C_0 + C_1 z + C_2 z^2 + \cdots \\ &= U(r,\theta)+iV(r,\theta) \end{aligned} \tag{6}$$

which is evidently continuous in the closed unit circle, must have the property

$$V(1,\theta)=V(\theta) \geqq 0 \quad \text{for} \quad 0 \leqq \theta \leqq 2\pi \tag{7}$$

if ε denotes $+1$ or -1 according as $f(z)$ satisfies the condition (3) or (4).

By the actual multiplication of $F(z)$ and

$$\frac{1}{g(x)} = e^{i\frac{\sigma_1+\sigma_2}{2}} \times \frac{z}{(e^{i\sigma_1}-z)(e^{i\sigma_2}-z)} = \frac{1}{2i \sin \frac{\sigma_2-\sigma_1}{2}} \sum_{n=1}^{\infty} (e^{-in\sigma_1} - e^{-in\sigma_2}) z^n \tag{8}$$