109. *On the Distributivity of a Lattice of Lattice-congruences.*

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In a previous note[1] one of us studied the structure of the lattice formed of congruences of a finite-dimensional lattice to prove that it is a distributive lattice. In the following we want to show that the congruences of any lattice, not necessarily finite-dimensional, form always a distributive lattice. The proof is quite simple and direct. Namely:

Let L be a lattice and let $\Phi=\{\varphi\}$ be the (complete) lattice of its congruences; we mean by $\varphi_1 \geqq \varphi_2$ that[2] $a \equiv b$ mod. φ_1 implies $a \equiv b$ mod. φ_2. Thus $a \equiv b$ mod. $\varphi_1 \cup \varphi_2$ when and only when a and b are congruent both mod. φ_1 and mod. φ_2, while $a \equiv b$ mod. $\varphi_1 \cap \varphi_2$ is equivalent to that there exists a finite system of elements $c_1, c_2, \dots, c_n$ in L such that

$$(1) \qquad a \equiv c_1(\varphi_1),\ c_1 \equiv c_2(\varphi_2),\ c_2 \equiv c_3(\varphi_1), \dots,\ c_{n-1} \equiv c_n(\varphi_1),\ c_n \equiv b(\varphi_2).$$

Consider arbitrary three congruences φ_1, φ_2 and φ_3. Obviously $(\varphi_1 \cap \varphi_2) \cup \varphi_3 \leqq (\varphi_1 \cup \varphi_3) \cap (\varphi_2 \cup \varphi_3)$. In order to prove the converse inclusion, assume

$$(2) \qquad a \equiv b \text{ mod. } (\varphi_1 \cap \varphi_2) \cup \varphi_3$$

for a certain pair $a > b$ of elements in L. Then $a \equiv b$ mod. φ_3 and there is a finite set of elements $c_1, c_2, \dots, c_n$ such that (1) holds. Now, the transformation

$$x \rightarrow x' = (x \cap a) \cup b$$

maps L onto the interval $[b, a]$, and it preserves any congruence relation. On applying this tranformation to (1), we see that we may assume without loss of generality that

$$a \geqq c_i \geqq b \qquad (i=1, 2, \dots, n).$$

But then, since $a \equiv b$ mod. φ_3, the elements a, b and c_i are all congruent mod. φ_3. Hence

$$a \equiv c_1(\varphi_1 \cup \varphi_3),\ c_1 \equiv c_2(\varphi_2 \cup \varphi_3),\ c_2 \equiv c_4(\varphi_1 \cup \varphi_3), \dots$$
$$\dots,\ c_{n-1} \equiv c_n (\varphi_1 \cup \varphi_3),\ c_n \equiv b(\varphi_2 \cup \varphi_3),$$

which means

$$a \equiv b \text{ mod. } (\varphi_1 \cup \varphi_3) \cap (\varphi_2 \cup \varphi_3).$$

Since this is the case for every pair $a > b$ in L satisfying (2), we have $(\varphi_1 \cap \varphi_2) \cup \varphi_3 \geqq (\varphi_1 \cup \varphi_3) \cap (\varphi_2 \cup \varphi_3)$ as desired. Thus

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1) N. Funayama, On lattice congruence, Proc. **18** (1942).

2) Contrary to the previous note, l. c. 1).