107. An Abstract Integral (X).

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Introduction. The first section is devoted to simplify the theory of general Denjoy integral. The essential point is to use Romanowski's lemma[1]. He used the lemma to develop the theory of the special Denjoy integral in abstract space. In § 2 we define an "abstract Denjoy integral" The integral which is called $\mathfrak{D}$-integral, becomes general or special Denjoy integral and others by the suitable specialization. The ($\mathfrak{D}$)-integral is defined as the inverse of an "abstract derivative "$\mathfrak{A}\mathfrak{D}$ which is defined axiomatically. Finally, we remark that the theory developed here can be extended to the case of abstract valued functions defined in an abstract space.

§ **1.** Let $f(x)$ be a real valued function in the interval $I_0=(a, b)$. If $f(x)$ is a continuous function in I_0 such that there is a sequence of sets (E_k) such as $I_0=\bigvee_{k=1}^{\infty} E_k$ and $f(x)$ is absolutely continuous in E_k $(k=1, 2, 3, \dots)$, then $f(x)$ is called to be generalized absolutely continuous in I_0, and we write $f\varepsilon CAC_{I_0}$ or simply $f\varepsilon GAC$. Approximate derivative $ADF(x)$ of $f(x)$ is defined in the ordinary manner.

We will begin by two lemmas.

(1.1) Let E be a closed set in I_0 and $I_0=\bigvee_{k=1}^{\infty} E_k$, then there is a portion P of E such that a suitable E_k is dense in P.

Proof. If the theorem is not true, then there is a portion P_1 of E such that $P_1 \frown E_1=\theta$. There is also a portion P_2 of P_1 such that $P_2 \frown E_2=\theta$. Thus proceeding we get a sequence (P_k) of portions such that $P_k \geqq P_{k+1}$ $(k=1, 2, 3, \dots)$. Evidently $\bigwedge_{k=1}^{\infty} P_k \neq \theta$. If $x\varepsilon \bigwedge_{k=1}^{\infty} P_k$, then $x\varepsilon E$. On the other hand $x\bar{\varepsilon} E_k$ $(k=1, 2, 3, \dots)$, and then $x\bar{\varepsilon} I_0$ which is a contradiction.

(1.2) (Romanowski) Let $\mathfrak{J}$ be a system of open intervals in I_0, such that[1]

1°. $I_k\varepsilon\mathfrak{J}$ $(k=1, 2, \dots, n)$ and $(\bigvee_{k=1}^{\infty} \bar{I}_k)^0=I$ imply $I\varepsilon\mathfrak{J}$.

2°. $I\varepsilon\mathfrak{J}$ and $\mathfrak{J}' \leqq I$ imply $I'\varepsilon\mathfrak{J}$.

3°. if $\bar{I}' \leqq I$ implies $I'\varepsilon\mathfrak{J}$, then $I\varepsilon\mathfrak{J}$.

4°. if I_1 is a subsystem of $\mathfrak{J}$ such that $\mathfrak{J}_1$ does not cover I_0, then there is an $I\varepsilon\mathfrak{J}$ such that $\mathfrak{J}_1$ does not cover I.

Then $I_0\varepsilon\mathfrak{J}$.

Proof. 4° implies $\bigvee(I;\ I\varepsilon\mathfrak{J}) \geqq I_0$. Let $\bar{I} \subset I_0$. By the Heine-Borel theorem there are I_k $(k=1, 2, \dots, n)$ in $\mathfrak{J}$ such as $I \subset \bigvee_{k=1}^{n} I_k$. End

1) Romanowski, Recueil math., 1940.