106. *An Abstract Integral (IX).*

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Methods defining integral without use of measure was studied by W. H. Young, P. J. Daniel[1], S. Banach[2], and H. H. Goldsteine[3], S. Izumi[4] extended Banach's method to the case of vector lattice. Many authors defined Lebesgue integral as an extension of Riemann or "abstract" Riemann integral. In this paper, extending Goldsteine's method we give a "Lebesgue integral" as an extension of a certain non-negative functional on a vector lattice.

§ **1.** Let Z be a vector lattice, and X a sublattice of Z which has the following properties: for any $z\in Z$ there exists at least one $x\in X$ such as $|z|\leqq x$. Let $f(x)$ be a linear non-negative functional on the domain X.

If we define the functionals

$$f^0(z)\equiv \underset{z\leqq x\in X}{\text{gr. l. b}}\, f(x)\,, \qquad f_0(z)\equiv \underset{z\geqq x\in X}{\text{l. u. b}}\, f(x)$$

on Z, then we have

(1.1) $f_0(z)\leqq -f^0(-z)$.

(1.2) $f_0(z)\leqq f^0(z)$.

(1.3) $f_0(z_1+z_2)\geqq f_0(z_1)+f_0(z_2)$, $f^0(z_1+z_2)\leqq f^0(z_1)+f^0(z_2)$.

(1.4) $f_0(cz)=cf_0(z)$ and $f^0(cz)=cf^0(z)$, for any real non-negative number c.

(1.5) $f_0(z_1)+f_0(z_2)\leqq f_0(z_1\wedge z_2)+f_0(z_1\vee z_2)\leqq f^0(z_1\wedge z_2)$
$+f^0(z_1\vee z_2)\leqq f^0(z_1)+f^0(z_2)$.

(1.6) $f^0(|z|)-f_0(|z|)\leqq f^0(z)-f_0(z)$.

We shall prove the last two only. If $x_1\geqq z_1$, $x_2\geqq z_2$, $x_1\in X$, $x_2\in X$, then we have $x_1\vee x_2\geqq z_1\vee z_2$ and $x_1\wedge x_2\geqq z_1\wedge z_2$, and then

$$f^0(z_1\wedge z_2)+f^0(z_1\vee z_2)\leqq f(x_1\wedge x_2)+f(x_1\vee x_2)=f(x_1+x_2)=f(x_1)+f(x_2)\,,$$

$$f^0(z_1\wedge z_2)+f^0(z_1\vee z_2)\leqq f^0(z_1)+f^0(z_2)\,.$$

Similarly $\qquad f_0(z_1\wedge z_2)+f_0(z_1\vee z_2)\geqq f_0(z_1)+f_0(z_2)$.

Hence we have the relation (1.5). In the next place by

$$|z|=z\vee(-z)\,,\quad\text{and}\quad -|z|=(-z)\wedge z\,,$$

1) Annals of Mathematics, **19** (1918).
2) S. Saks: The theory of integral.
3) Bull. of the Amer. Math. Soc., **47** (1941).
4) S. Izumi: Isōsūgaku 3-2, (1941).