# 105. *An Abstract Integral (VIII).*

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*Introduction.* In this paper we intend to establish the theory of Lebesgue integral of the vector lattice valued functions. This subject has been discussed by Bochner[1] and Izumi[2]. Our consideration differs from them in that it is based on the notion of semi-ordering.

We define the Lebesgue integral which is analogous to Young[3], Daniell[4], and Banach's[5] one in real valued functions. It is noteworthy that the integrable functions are not always approximated by step functions or Riemann integrable functions, although the integral is obtained by an extension from step functions or Riemann integrable functions. This integral includes obviously the Bochner's and, if we neglect conditions on the vector lattice, includes the Izumi's.

And moreover our considerations can be abstracted in that way which regards the extension of an integral as the extension of a linear operation between two given vector lattices. This problem has been treated by Izumi and Nakamura[6] in the case of a linear functional.

**1.** *The class* $T_0$. Let $f(t)$ be an abstract function defined in abstract space and with range in a complete regular vector lattice $L$[7].

We assume that the initial class $T_0$ of functions is closed with respect to the operations:—$cf$, $f_1+f_2$, $f_1 \cup f_2$, $f_1 \frown f_2$, and that the functions of $T_0$ are bounded. Further let a functional operation $I(f)$ be defined on $T_0$ such that

(A) $I(f_1+f_2)=I(f_1)+I(f_2)$;

(L) If $f_1 \geqq f_2 \geqq \cdots$ and $\lim f_n=0$, then $\lim I(f_n)=0$.

From these we can easily conclude that

(C) $I(cf)=cI(f)$, where $c$ is a real constant;

(P) If $f \geqq 0$, $I(f) \geqq 0$.

Then the class $T_0$ is obviously a lattice. For some instances of the class $T_0$, we may consider the class of step functions or Riemann integrable functions.

**2.** *Extension to class* $T_1$ *from* $T_0$. If $f_1 \leqq f_2 \leqq \cdots$ where $f_i \in T_0$, then $\lim f_n$ exists (if we adjoin $+\infty$ to the range), and we define $T_1$ as class of such limit functions. For such $(f_n)$ we have $I(f_1) \leqq I(f_2) \leqq \cdots$ and then $\lim I(f_n)$ exists (if allow $+\infty$ as limit).

1) Bochner, Nat. Acad. Sci., **26** (1940), p. 29.
2) Izumi, Proc. **18** (1942), 53.
3) Young, Proc. London Math. Soc., **18** (1914), p. 109.
4) Daniell, Ann. of Math., **19** (1917), p. 279.
5) Saks, *Theory of the Integral*, 1937, p. 320.
6) Izumi and Nakamura, Proc. **16** (1940), 518.
7) For these definitions and discussions, see Kantorovitch, Recueil Math. of Moskau, **49** (1940), p. 209, and Orihara, This Proc. Regurality is used in (4.6) and (4.7) only.