# 124. On the Zeros of the Riemann Zeta-function.

By Masatsugu TSUJI.

Mathematical Institute, Tokyo Imperial University.

(Comm. by T. YOSIE, M.I.A., Dec. 12, 1942.)

Littlewood[1] proved that the Riemann zeta-function $\zeta(s)$ $(s=\sigma+it)$ has a zero in the domain: $0<\sigma<\infty$, $|t-T|<\frac{16}{\log\log\log T}(T\geqq T_0)$. Simple proofs are given by Hoheisel[2], Titchmarsh[3] and Kramaschke[4]. These authors use the Hadamard's three circles theorem in the proof. I will here give a still simpler proof, where I use the Doetsch's three lines theorem[5] in the modified form.

*Theorem.* *$\zeta(s)$ has a zero in the domain: $0<\sigma<\infty$, $|t-T|<\frac{\varkappa}{\log\log\log T}$ $\left(T\geqq T(\varkappa)\right)$, where $\varkappa$ is any positive number greater than $\frac{\pi}{4}$.*

Especially we may take $\varkappa=1$.

First we will prove a lemma.

*Lemma.* *Let $f(z)$ be regular and bounded in $|z|<1$ and $K(r)$ be a circle: $|z-(1-r)|=r$ $(0<r\leqq 1)$ and $M(r)=\underset{z \text{ on } K(r)}{\text{Max.}}|f(z)|$. Then*

$$M(r_2)\leqq M(r_1)^{\frac{\frac{1}{r_2}-\frac{1}{r_3}}{\frac{1}{r_1}-\frac{1}{r_3}}}M(r_3)^{\frac{\frac{1}{r_1}-\frac{1}{r_2}}{\frac{1}{r_1}-\frac{1}{r_3}}}\qquad (0<r_1<r_2<r_3\leqq 1).$$

*Proof.* By $s=\frac{1}{1-z}$, we map $|z|<1$ on the half-plane $\Re(s)>\frac{1}{2}$, then $K(r)$ becomes a line $\Re(s)=\frac{1}{2r}$, so that the lemma follows from the Doetsch's three lines theorem[5].

*Proof of the theorem.*

Suppose that $\zeta(s)$ has no zero in the domain $\varDelta$: $0<\sigma<\infty$, $|t-T|<c=\frac{\varkappa}{\log\log\log T}$ $\left(\varkappa>\frac{\pi}{4}\right)$, then $\log\zeta(s)$ is regular in $\varDelta$. We map $\varDelta$ on $|z|<1$ by

---

1) Littlewood: Two notes on the Riemann zeta-function. Proc. Cambridge Phil. Soc. **22** (1924).

2) Hoheisel: Jahresbericht Schles. Ges. vaterl. Kultur **99**.

3) Titchmarsh: On the Riemann zeta-function. Proc. Cambridge Phil. Soc. **28** (1932).

4) Kramaschke: Nullstellen der Zetafunktion. Deutsche Math. **2** (1937).

5) Doetsch: Über die obere Grenze des absoluten Betrages einer analytischen Funktion auf Geraden. Math. Z. **8** (1920).