# 122. On some Properties of Hausdorff's Measure and the Concept of Capacity in Generalized Potentials.

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## I. Hausdorff's measure and upper density.

Let $\Omega$ be any *separable* metric space with the distance $\rho(p, q)$ for $p, q \in \Omega$.

A *sphere* in $\Omega$ with radius $r$, of centre $a$ is the set of points $p$ such that $\rho(p, a) < r$.

Given any set $E \subset \Omega$. let $\delta(E)$ be the diameter of $E$, that is, $\delta(E) = \sup_{p,q \in E} \rho(p, q)$.

Now, let $h(r)$ be a positive, continuous, monotone-increasing function defined for $r > 0$ near the origin such that

$$\lim_{r \to 0} h(r) = 0 .$$

Taking any sequence of spheres $\{S_i\}_{i=1,2,\dots}$ such that

$$\text{(i)} \quad \sum_{i=1}^{\infty} S_i \supset E , \qquad \text{(ii)} \quad \delta(S_i) < \varepsilon \ (i = 1, 2, \dots) ,$$

let us put $m_h(E, \varepsilon) = \inf \sum_{i=1}^{\infty} h[\delta(S_i)]$ for fixed $\varepsilon > 0$, and write $m_h(E) = \lim_{\varepsilon \to 0} m_h(E, \varepsilon)$ which is called $h$-measure of $E$. In this definition, we may assume, without loss of generality, that each $S_i$ has points common with $E$. This measure, introduced first by F. Hausdorff[1] is known to have the property of Carathéodory's outer measure and therefore the measurable class of sets with respect to the $h$-measure contains all the Borel sets.

Moreover, $h$-measure is a *regular measure*[2], that is to say, for any set $E \subset \Omega$, there exists a Borel set, $H \in \mathfrak{G}\delta$, such that $H \supset E$ and $m_h(H) = m_h(E)$.

If $\Omega$ is 2-dimensional Euclidean space, $m_h(E)$ for $h(r) = \frac{\pi}{4} r^2$, $h(r) = r^\alpha$ $(\alpha > 0)$ and $h(r) = \left(\log \frac{1}{r}\right)^{-1}$ are Lebesgue's plane measure, $\alpha$-dimensional measure (if $\alpha = 1$, then called Carathéodory's linear measure or length of $E$) and logarithmic measure respectively.

Given a set $E$ and a point $p \in \Omega$, we shall define the *upper density* of $E$ at $p$ with respect to the $h$-measure by the following expression:

$$\Delta_h(p, E) = \overline{\lim_{\delta(S) \to 0}} \frac{m_h(E \cdot S)}{h[\delta(S)]} ,$$

1) F. Hausdorff. Dimension und äusseres Mass, Math. Annalen., **78** (1919).
2) F. Hausdorff. Loc. cit.