117. Locally Bounded Linear Topological Spaces.

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D. H. Hyers[1] has introduced the notion of absolute value into locally bounded linear topological spaces, and proved that the absolute value is upper semi-continuous, while J. v. Wehausen[2] showed that a linear topological space is metrizable as an F-metric if and only if it satisfies the first countability axiom. Since every locally bounded linear topological space satisfies the first countability axiom, it is metrizable as an F-metric. But all F-metric spaces are not necessarily locally bounded. Hence the problem arises: what metric spaces are equivalent to locally bounded linear topological spaces?

In this paper we introduce a lower or upper semi-continuous absolute value into locally bounded linear topological space and give a condition that the absolute value is continuous. We define F'-normed spaces and prove that they are equivalent to locally bounded linear topological spaces.

§ 1. *Definitions and lemmas.*

Definition 1. A linear space L is called a linear topological space if there exists a family $\mathfrak{U}$ of sets $U\subset L$ satisfying following conditions[3] ·

1) *The intersection of all the sets $\in\mathfrak{U}$ is $\{\theta\}$.*[1]
2) *If $U, V\in\mathfrak{U}$ there exists $W\in\mathfrak{U}$ such that $W\subset U\frown V$.*
3) *If $U\in\mathfrak{U}$ there exists $V\in\mathfrak{U}$ such that $V+V\subset U$.*[2]
4) *If $U\in\mathfrak{U}$ there exists $V\in\mathfrak{U}$ such that $[-1,1]V\subset U$.*[3]
5) *If $x\in L$, $U\in\mathfrak{U}$ there exists real α such that $x\in\alpha U$.*

Definition 2. A linear topological space L is called locally bounded if $\mathfrak{U}$ satisfies:

6) *There exists a bounded set*[4] *V of $\mathfrak{U}$.*

Lemma 1. If we put $H=[-1,1]V$, then

1) *$[-1,1]H=H$.*
2) *$0<\alpha<\beta$ implies $\alpha H\subset\beta H$.*
3) *H is bounded.*
4) *For every $\alpha,\beta\geqq 0$, $\alpha+\beta=1$ there exists a constant $k\geqq 1$ independent of α,β such that $\alpha H+\beta H\subset kH$.*
5) *Let $\mathfrak{U}^*=\{\alpha H\}$, $\alpha>0$. Then $\mathfrak{U}^*$ is equivalent to $\mathfrak{U}$.*

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1) $\{\theta\}$ is the set consisting of zero element θ only.

2) If $S, T\subset L$, $S+T$ is the set of all $x+y$, where $x\in S$, $y\in T$.

3) $[-1,1]V$ is the set of all αx such as $-1\leqq\alpha\leqq 1$, $x\in V$.

4) A set S in a linear topological space will be called bounded if for any $U\in\mathfrak{U}$ there is a number α such as $S\subset\alpha U$. (v. Neumann) This is the same to say that for and sequence $\{x_n\}\subset S$ and any real sequence $\{\alpha_n\}$ converging to 0, the sequence $\{\alpha_n x_n\}$ converges to θ. (Banach and Kolmogoroff)