14. On the Uniform Distribution of Values of a Function mod. 1.

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1. *Uniform distribution of values of $f(x)$ mod. 1.*

Let $f(x)$ be a continuous function defined for $0 \leqq x < \infty$ and $(f(x)) = f(x) - [f(x)]$, so that $0 \leqq (f(x)) < 1$. Let $\mathfrak{A} = [\alpha, \beta]$ $(0 \leqq \alpha < \beta \leqq 1)$ be an interval in $[0, 1]$ and $E(r, \mathfrak{A})$ be the set of points x on the x-axis, which lie in $[0, r]$, such that $\alpha \leqq (f(x)) \leqq \beta$ and $mE(r, \mathfrak{A})$ be its measure. If for any $\mathfrak{A}$,

$$\lim_{r\to\infty} \frac{mE(r, \mathfrak{A})}{r} = |\mathfrak{A}| = \beta - \alpha, \tag{1}$$

then we say that the values of $f(x)$ distribute uniformly mod. 1.

H. Weyl[1] proved that (I) the necessary and sufficient condition, that the values of $f(x)$ distribute uniformly mod. 1 is that

$$\int_0^r e^{2\pi a i f(x)} dx = o(r), \tag{2}$$

for any integer $a\ (\neq 0)$.

(II) Let $F(t)$ be periodic with period 1 and be integrable in Riemann's sense in $[0, 1]$. If the values of $f(x)$ distribute uniformly mod. 1, then

$$\lim_{r\to\infty} \frac{1}{r} \int_0^r F\big(f(x)\big) dx = \int_0^1 F(t) dt. \tag{3}$$

We will prove

Theorem I. Let $f(x)$ be a positive continuous increasing convex function of $\log x$, *such that* $\lim_{x\to\infty} \frac{f(x)}{\log x} = \infty$, *then the values of $f(x)$ distribute uniformly mod. 1.*

Proof. Let $a\ (\neq 0)$ be an integer and put $t = 2\pi a f(x) = \varphi(x)$ and $x = \psi(t)$ be its inverse function. We suppose that $a > 0$; the case $a < 0$ can be proved similarly. From the convexity of $f(x)$ as a function of $\log x$, $x\varphi'(x) = \frac{\psi(t)}{\psi'(t)}$ is an increasing function[2] of x. If $x\varphi'(x) < K$ for $0 \leqq x < \infty$, then $\varphi(x) = O(\log x)$, which contradicts the hypothesis. Hence $\lim_{x\to\infty} x\varphi'(x) = \infty$, so that $\frac{\psi'(t)}{\psi(t)}$ is a decreasing function of t and

1) H. Weyl: Über die Gleichverteilung von Zahlen mod. 1. Math. Ann. **77** (1916). In Weyl's paper (II) is not expressed explicitly, but (II) follows from (I) easily.

(2) $\varphi'(x)$ may cease to exist at an enumerable set of points, where we define $\varphi'(x)$ suitably.