32. Notes on Infinite Product Measure Spaces, I.

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The purpose of this note is to give a simple and new proof to the existence of an independent product measure on a Cartesian infinite product space.

Let $\{(\Omega^\gamma, \mathfrak{B}^\gamma, m^\gamma) \mid \gamma \in \Gamma\}$ be a family of measure spaces satisfying $m^\gamma(\Omega^\gamma)=1$ for each $\gamma \in \Gamma$, where we mean by a *measure space* $(\Omega, \mathfrak{B}, m)$ a triple of a space Ω (without topology), a Borel field $\mathfrak{B}$ of subsets B of Ω, and a countably additive measure $m(B)$ defined on $\mathfrak{B}$ (with $0 < m(\Omega) < \infty$). We shall first define a measure space $(\Omega^*, \mathfrak{B}^*, m^*)$ which we call the *independent product measure space* of the family $\{(\Omega^\gamma, \mathfrak{B}^\gamma, m^\gamma) \mid \gamma \in \Gamma\}$.

The space Ω^*, which is symbolically denoted as

(1) $$\Omega^* = \mathbf{P}_{\gamma \in \Gamma} \Omega^\gamma$$

is the set of all Γ-sequences (or functions defined on Γ)

(2) $$\omega^* = \{\omega^\gamma \mid \gamma \in \Gamma\}$$

such that $\omega^\gamma \in \Omega^\gamma$ for each $\gamma \in \Gamma$.

A subset R^* of Ω^* is called *rectangular* if it is of the form:

(3) $$R^* = B^{\gamma_1} \times \cdots \times B^{\gamma_n} \times \mathbf{P}_{\gamma \in \Gamma - \{\gamma_1, \dots, \gamma_n\}} \Omega^\gamma$$

where $B^{\gamma_i} \in \mathfrak{B}^{\gamma_i}$, $i=1, \dots, n$, and $\{\gamma_1, \dots, \gamma_n\}$ is an arbitrary finite system of elements from Γ. R^* is, by definition, the set of all $\omega^* = \{\omega^\gamma \mid \gamma \in \Gamma\} \in \Omega^*$ such that $\omega^{\gamma_i} \in B^{\gamma_i}$ for $i=1, \dots, n$. The family of all rectangular sets R^* of Ω^* is denoted by $\mathfrak{R}^*$.

Further, a subset E^* of Ω^* is called *elementary* if it is of the form:

(4) $$E^* = \mathbf{U}_{i=1}^n R_i^*$$

where $R_i^* \in \mathfrak{R}^*$ for $i=1, \dots, n$. We may assume that the R_i^* in (4) are mutually disjoint. This follows from the fact that the intersection of two rectangular set of Ω^* is again rectangular, and that the complementary of a rectangular set of Ω^* is expressible as the union of a finite number of mutually disjoint rectangular sets of Ω^*. The family of all elementary sets E^* of Ω^* is denoted by $\mathfrak{E}^*$. It is clear that $\mathfrak{E}^*$ is a field.

We shall next define a set function $m^*(R^*)$ on $\mathfrak{R}^*$ by

(5) $$m^*(R^*) = m^{\gamma_1}(B^{\gamma_1}) \times \cdots \times m^{\gamma_n}(B^{\gamma_n})$$

if R^* is of the form (3), and then $m^*(E^*)$ on $\mathfrak{E}^*$ by

(6) $$m^*(E^*) = \sum_{i=1}^n m^*(R_i^*)$$