# 27. On a Characterisation of Order-preserving Mapping-lattice.

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**1.** *Introduction.* A mapping $f$ of a lattice $L_1$ into a lattice $L_2$ is called order preserving, when for any two elements $a > b$ of $L_1$, there holds the relation $f(a) > f(b)$ in the order of $L_2$[1]. If we define $f_1 > f_2$, when for any element $a$ of $L_1$ $f_1(a) > f_2(a)$ is satisfied, then the set of all order preserving mappings forms a lattice $\{f\}$. The join $f_1 \cup f_2$ and the meet $f_1 \cap f_2$ are respectively defined by the following mappings:

$$(f_1 \cup f_2)(a) = f_1(a) \cup f_2(a),$$
$$(f_1 \cap f_2)(a) = f_1(a) \cap f_2(a).$$

In this paper we are concerned with the problem of a lattice-theoretic characterisation of this order preserving transformation-lattice for the case, when $L_2$ is the two-element lattice $\{0, 1\}$.

The lattice $L^*$ in the theorem of this paper is isomorphic with the ring of all $M$-closed subsets of the lattice $L$ of its join-irreducible elements. Evidently we can generalise the theorem for the case, when $L$ is only a partially ordered set in the order of $L^*$. In this case we can therefore omit the condition (iv) of the theorem[2]. When $L_1$ is a Boolean algebra, i. e. the lattice of all subsets of a set $R$, whose order relation is defined by the inclusion relation as usual, then the mapping-lattice is the same as the covering lattice of all subsets of $R$.

**2.** *Transformation-lattice.*

Lemma 1. All order preserving mappings $\{f\}$ of a lattice $L$ into the lattice $\{0, 1\}$ form a complete and complete distributive lattice.

Proof. For any subset $\{f_x \mid x \in X\}$ of $\{f\}$ and for any element a of $L$ we have the relations;

$$(\bigcup_{x|X} f_x)(a) = \bigcup_{x|X}\big(f_x(a)\big),$$
$$(\bigcap_{x|X} f_x)(a) = \bigcap_{x|X}\big(f_x(a)\big).$$

Furthermore for one element $f_0 \in \{f\}$ we can easily prove

$$\begin{aligned}\big(f_0 \cup (\bigcap_{x|X} f_x)\big)(a) &= f_0(a) \cup (\bigcap_{x|X} f_x)(a)\\ &= f_0(a) \cup \big(\bigcap_{x|X} f_x(a)\big)\\ &= \bigcap_{x|X}\big((f_0(a) \cup f_x(a)\big) = \bigcap_{x|X}\big((f_0 \cup f_x)(a)\big),\end{aligned}$$

1) We use the symbol $>$ in the meaning of the usual symbol $\geqq$.
2) See Birkhoff: Lattice Theory, p. 76.