26. *On a Characterisation of Join Homomorphic Transformation-lattice*

By Atuo KOMATU.
Mathematical Institute, Osaka Imperial University.
(Comm. by T. TAKAGI, M.I.A., March 12, 1943.)

1. *Introduction.* A mapping f of a lattice L_1 into a lattice L_2 is called join homomorphic, when for any elements a, b of L_1 there exists the relation

$$f(a \cup b) = f(a) \cup f(b).$$

This mapping is order preserving, for, if $a > b$ in L_1, it follows $f(a) = f(a \cup b) = f(a) \cup f(b)$, i.e. $f(a) > f(b)$ in L_2.

If we define $f_1 > f_2$, when for any element a of L_1 $f_1(a) > f_2(a)$ is satisfied, then the set of all join homomorphic transformations forms a partially ordered set $\{f\}$. If L_2 is complete and completely distributive, then $\{f\}$ is a complete lattice. For there exist the following relations for any element a of L_1

$$(f_1 \cup f_2)(a) = f_1(a) \cup f_2(a),$$

$$\left(\bigcup_X (f_x \mid X)\right)(a) = \bigcup_X \left(f_x(a) \mid X\right),$$

$$(f_1 \cap f_2)(a) = \bigcup_X \left(g_x(a) \mid X\right),$$

$$\left(\bigcap_X (f_x \mid X)\right)(a) = \bigcup_Y \left(h_y(a) \mid Y\right),$$

where $\{g_x \mid x \in X\}$ is the set of all transformations such that $g_x < f_1, f_2$, and $\{h_y \mid y \in Y\}$ is the set of all transformations such that $h_y < f_x$ for all x of X. This join $f_1 \cup f_2$, meet $f_1 \cap f_2$, complete join $\bigcup_X f_x$ and complete meet $\bigcap_X f_x$ are again clearly join homomorphic transformations.

In this paper we are concerned with the problem of a lattice-theoretic characterisation of this join homomorphic transformation-lattice for the case, when L_2 is the two-element lattice $\{0, 1\}$.

Lemma 1. All ideals in L form a lattice, which is dual isomorphic with the join homomorphic transformation-lattice $\{f\}$ of L into $\{0, 1\}$.

Proof. Let f be a join homomorphic mapping of L into $\{0, 1\}$. Then the set $f^{-1}(0)$ is an ideal in L. For if $a, b \in f^{-1}(0)$, then $f(a \cup b) = f(a) \cup f(b) = 0$; therefore $a \cup b \in f^{-1}(0)$. And if $a \in f^{-1}(0)$, $b < a$, then clearly $f(b) < f(a) = 0$. Hence $f^{-1}(0)$ includes b.

Conversely, let $\mathfrak{A}$ be an ideal in L, then the transformation f such that

$$f(a) = 0, \qquad a \in \mathfrak{A},$$

$$f(a) = 1, \qquad a \notin \mathfrak{A},$$

1) Cf. A. Komatu. On a Characterisation of Order Preserving Transformation-lattice. Proc. **19** (1943), 27.