23. Notes on Differentiation.

By Shin-ichi IZUMI and Gen-ichirô SUNOUCHI.
Mathematical Institute, Tohoku Imperial University, Sendai.
(Comm. by M. FUJIWARA, M.I.A., March 12, 1943.)

Important theorems concerning differentiation are devided into two classes. The first class consists of theorems of differentiability of indefinite integrals and related theorems. The second is the class of Denjoy's theorem and its analogue. We will give a universal method to prove theorems of the first class, and prove a convergence theorem which contains theorems of the second class. Our method is to use maximal theorem due to Hardy and Littlewood and convergence theorem due to Kantorovitch. This idea is due to K. Yoshida[1] and Kantorovitch[2].

1. *Theorems of Kantorovitch and Hardy-Littlewood.*

Kantorovitch's theorem reads as follows[3].

(*K*) *Let X and Y be regular vector lattices and (U_n) be a sequence of operations from X to Y such that $U_n \in H_t^t$ $(n=1,2,3,\dots)$ (by the Kantorovitch's notation). If*

1°. *$U_n(x)$ converges in a dense set D in X,*

2°. *for any x in X $\limsup U_n(x)$ and $\liminf U_n(x)$ exists, then $U_n(x)$ (o)-converges for all x in X.*

lim sup and lim inf denote those concerning order topology. If $Y=S$, then the order limit becomes almost everywhere convergence.

On the other hand maximal theorem reads as follows[3].

(*HL*) *We put $y(s) \equiv \sup\left(\frac{1}{|I|}\int_I x(t)dt;\ s \in I\right)$ for integrable function $x(t)$. Then*

1°. *If $x \in L^p$ $(p>1)$, then $y \in L^p$ and $\int_0^1 |y(t)|^p dt \leqq A\int_0^1 |x(t)|^p dt$.*

2°. *If $x \in L_Z$, then $y \in L$ and $\int_0^1 |y(t)|\, dt \leqq A\int_0^1 |x(t)| \log^+ |x(t)|\, dt + B$,*

3°. *If $x \in L$, then $y \in L^\alpha$ $(0<\alpha<1)$ and $\left(\int_0^1 |y(t)|^\alpha\right)^{\frac{1}{\alpha}} dt \leqq A\int_0^1 |x(t)|\, dt$,*

where A and B are independent of function $x(t)$, and L_Z denotes the Zygmund class.

The last is due to Privaloff, which is generalized as follows.

3^{∞}. *If $x \in L$, then $y \in L_K$, that is, there exists the integral*

---

1) Yosida's result was not yet published.

2) Kantorovitch, Comptes Rendus Acad. Sci. URSS., **14** (1937), 225 and **14** (1937), 244.

3) Hardy-Littlewood, Acta Math., **54** (1930), 81. See Zygmund, *Trigonometrical Series*, (1935), 150.