22. A Remark on Ergodic Theorems.

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1. G. D. Birkhoff proved the following theorems.

(*B. 1*) Let T be a measure preserving transformation in $(0,1)$ such that the inverse transformation T^{-1} is also. Then for any $x=x(t)$ in $L=L(0,1)$ the limit

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{n=0}^{N}x(T^{n}t) \tag{1}$$

exists almost everywhere.

(*B, 2*) Let $T^{\lambda}(-\infty<\lambda<\infty)$ be a set of transformations satisfying above condition such that $T^{\lambda}(T^{\mu}t)=T^{\lambda+\mu}t$. If $x(T^{\lambda}t)$ is measurable in the product space (λ, t) and is integrable in $(0,1)$ with respect to t, then the limit

$$\lim_{N\to\infty}\frac{1}{N}\int_{0}^{N}x(T^{\lambda}t)d\lambda \tag{2}$$

exists almost everywhere.

These are called individual ergodic theorems. Convergence in (1) and (2) is not dominated by integrable functions in general. But Fukamiya and Wiener proved that

(*FW*) If T $\big($or $T^{\lambda}(-\infty<\lambda<\infty)\big)$ satisfies above condition and $x(t)\varepsilon L_Z$, that is, $x(t)\log^{+}|x(t)|$ is integrable, then (1) $\big($or (2)$\big)$ converges dominated by integrable functions almost everywhere.

This is called dominated ergodic theorem. To prove above three theorems Wiener proved the fundamental lemma:

(*W*) Let $x(t)$ be a non-negative integrable function and

$$x^{*}(t)=\underset{0<N<\infty}{\text{l. u. b.}}\frac{1}{N+1}\sum_{n=0}^{N}x(T^{n}t)\qquad \Big(\text{or}=\underset{0<N<\infty}{\text{l. u. b.}}\frac{1}{N}\int_{0}^{N}x(T^{\lambda}t)d\lambda\Big)$$

then we have for any $\alpha>0$

$$\big((t;\,x^{*}(t)>\alpha\big)\leqq\frac{1}{\alpha}\int_{0}^{1}x(t)dt\,.$$

2. In order to prove (B, 1), (B, 2) Wiener proved the mean ergodic theorem in L. But we can prove them directly by using a convergence theorem due to Kantorovitch. Kantorovitch's theorem reads as follows.

(*K*) Let X and Y be regular vector lattices and $\{U_n(x)\}$ be a sequence of (t,t)-continuous operations from X to Y. Then if

1°. for x in a dense set D in X $U_n(x)$ is (o)-$\big($or (t)-$\big)$ convergent,