# PAPERS COMMUNICATED

## 21. *Notes on Banach Space (V): Compactness of Function Spaces.*

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**1.** We have proved[1] already

*Theorem 1.* A set $\mathfrak{F}$ in $(C)$ (=family of continuous functions in $(0,1)$) is compact when and only when

1°. $\mathfrak{F}$ is uniformly bounded,

2°. $\lim_{\delta\to 0}\frac{1}{\delta}\int_0^\delta f(x+t)dt=f(x)$ uniformly for all $x$ in $(0,1)$ and for all $f$ in $\mathfrak{F}$,

*Theorem 2.* A set $\mathfrak{F}$ in $(M)$ (=family of essentially bounded measurable functions in (0,1)) is compact when and only when 1° and

3°. $\lim_{\delta\to 0}\frac{1}{\delta}\int_0^\delta f(x+t)dt=f(x)$ uniformly almost everywhere for all $x$ in $(0,1)$ and for all $f$ in $F$.

On the other hand Phillips[2] proved a compactness theorem in Banach space, whence he derived the Kolmogoroff-Tulajkoff theorem concerning compactness in $(L^p)$ $(p\geqq 1)$. The latter theorem reads as follows

*Theorem 3.* A set $\mathfrak{F}$ in $(L^p)$ $(p\geqq 1)$ (=family of measurable functions whose $p$-th power is integrable in $(0,1)$) is compact when and only when

4°. for $f$ in $\mathfrak{F}$ $\int_0^1|f(t)|^p dt$ is uniformly bounded,

5°. $\lim_{\delta\to 0}\frac{1}{\delta}\int_0^\delta f(x+t)dt=f(x)$ uniformly in the $L^p$-mean.

Concerning space $(S)$ we proved in § 3

*Theorem 4.* A set $\mathfrak{F}$ in $(S)$ (=family of measurable functions in $(0,1)$) is compact when and only when

6°. $\operatorname{asy\cdot lim}_{(\delta,N)}\frac{1}{\delta}\int_0^\delta \big(f(x+t)\big)^N dt=f(x)$ uniformly for $f$ in $\mathfrak{F}$, where $\operatorname{asy\cdot lim}_{(\delta,N)}$ is the Moore-Smith limit in measure and

$$\big(f(t)\big)^N=f(t) \quad \text{if} \quad |f(t)|\leqq N \quad \text{and} \quad =0, \quad \text{otherwise.}$$

Summing up above results we get

*Theorem 5.* A set $\mathfrak{F}$ in $E$ where $E$ is $(C)$, $(M)$, $(L^p)$ $(p\geqq 1)$ or $(S)$, is compact when and only when

7°. $\mathfrak{F}$ is bounded concerning metric in $E$,

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1) S. Izumi, Proc. **15** (1938).

2) R. Phillips, Trans. Am. Math. Soc., vol. **44** (1940).