47. On the Domain of Existence of an Implicit Function defined by an Integral Relation $G(x, y)=0$.

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1. *Theorems of Julia and Gross.*

Let $G(x, y)$ be an integral function with respect to x and y and $y(x)$ be an analytic function defined by $G(x, y)=0$ and F be its Riemann surface spread over the x-plane. Let E be a set of points on the x-plane, which are not covered by F. Evidently E is a closed set.

Julia[1] proved that E does not contain a continuum. If $y(x)$ is an algebroid function of order n, such that $A_0(x)y^n+A_1(x)y^{n+1}+\cdots+A_n(x)=0$, where $A_i(x)$ are integral functions of x, then F consists of n sheets and covers every point on the x-plane exactly n-times, where a branch point of F of order k is considered as covered k-times by F. We will prove

Theorem I. If $y(x)$ is not an algebroid function of x, then F covers any point on the x-plane infinitely many times, except a set of points of capacity zero.

In this paper "capacity" means "logarithmic capacity."

If we interchange x and y, we have

Let $G(x, y)$ be an integral function with respect to x and y and $y(x)$ be an analytic function defined by $G(x, y)=0$. If $y(x)$ does not satisfy a relation of the form: $A_0(y)x^n+A_1(y)x^{n+1}+\cdots+A_n(y)=0$, where $A_i(y)$ are integral functions of y, then $y(x)$ takes any value infinitely many times, except a set of values of capacity zero.

This is a generalization of Picard's theorem for a transcendental meromorphic function for $|x|<\infty$.

Julia's proof depends on the following

Gross' theorem[2]: Let $f(z)$ be one-valued and regular on the Riemann surface F, which does not cover a continuum. If $f(z)$ tends to zero, when z tends to any accessible boundary point of F, then $f(z)\equiv 0$.

We will first extend this Gross' theorem in the following way.

Theorem II. Let $f(z)$ be one-valued and meromorphic on a connected piece F of its Riemann surface, whose projection on the z-plane lies inside a Jordan curve C and F do not cover a closed set E of positive capacity, which lies with its boundary entirely inside C. If $f(z)$ tends to zero, when z tends to any accessible boundary point of F, whose projection on the z-plane lies inside C, except enumerably infinite number of such accessible boundary points, then $f(z)\equiv 0$.

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1) G. Julia: Sur le domaine d'existence d'une fonction implicite défine par une relation entière $G(x, y)=0$. Bull. Soc. Math. (1926).

2) W. Gross: Zur Theorie der Differentialgleichungen mit festen kritischen Punkten. Math. Ann. **78** (1918).